ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 81, No. 2 (March, 2013), 849–851

COMMENT ON IF YOU'RE SO SMART, WHY AREN'T YOU RICH? BELIEF SELECTION IN COMPLETE AND INCOMPLETE MARKETS

FILIPPO MASSARI
Washington University in St. Louis, St. Louis, MO 63130, U.S.A.

Econometrica, Vol. 81, No. 2 (March, 2013), 849–851

COMMENT ON IF YOU'RE SO SMART, WHY AREN'T YOU RICH? BELIEF SELECTION IN COMPLETE AND INCOMPLETE MARKETS

BY FILIPPO MASSARI[1]

MARKET SELECTION STUDIES the evolution of capital shares among traders with heterogeneous beliefs operating in a market. For the case of complete markets, allowing for heterogeneous discount factors and learning, Theorem 8 in Blume and Easley (2006; henceforward B–E) provides a sufficient condition for the wealth share of a trader to converge to 0 (vanish). I show by means of a counterexample that Theorem 8 is incorrect.

Intuitively, the counterexample can be summarized as follows. The economy has three traders and states are i.i.d. Two traders have fixed beliefs that are equally wrong, but in opposite directions. The third trader is a Bayesian learner. His prior beliefs put positive weight only on the two incorrect models. According to B–E's criterion for accuracy, the Bayesian trader makes, in every period, more accurate forecasts those that of the other traders. Yet his wealth share does not converge to 1. The result follows from these considerations. First, the wealth share evolution depends on the gains from actual realizations (likelihood) and not on the sum of potential gains (sum of per period conditional expected log-likelihoods). Second, the likelihood that a Bayesian posterior attaches to a sequence of realizations cannot be higher than the likelihood attached by the best model in its prior support.[2]

Theorem 8 appears late in the paper and the only result in B–E that relies on it is the last example in Section 3. The example in B–E becomes correct by modifying the beliefs in the example to $p_t^i(a) = \frac{e^{t^4+t} - e^t}{e^{t^4+t} - 1}$ and $p_t^j(a) = \frac{e^{t^4} - 1}{e^{t^4+t} - 1}$, as the same exact conclusions as in B–E can be derived using the condition of Proposition 3 in Sandroni (2000).

1. BLUME–EASLEY'S SUFFICIENT CONDITIONS TO VANISH

I follow the notation of B–E. The economy contains I traders indexed by i, j and S states. Σ is the σ-algebra that contains all the infinite sequences of realizations σ. p is the true probability, and for every probability q on Σ, $q_t(\sigma)$ denotes the marginal probability that q attaches to the first t realizations of path σ. $\mathcal{F}_t$ is the information partition at period-event σ_t. On path σ, $I_t^s(\sigma)$ is the indicator function of state s on period t, $c_t(\sigma)$ is consumption at period t, and β_i, $p^i(\sigma)$ denote, respectively, trader i's discount factor and beliefs. For each trader i, $Z_t^i := -\sum_{s\in S} I_t^s(\sigma) \log p^k(s|\mathcal{F}_{t-1})$ and $\bar{Z}_t^k(\sigma^{t-1}) := \sum_{\tau=1}^t E\{Z_\tau^k|\mathcal{F}_{\tau-1}\}$.

[1]I wish to thank Werner Ploberger, John Nachbar, Lawrence Blume, and David Easley.

[2]For a more detailed discussion, I refer to Massari (2012).

 DOI: 10.3982/ECTA10455

Therefore, given a path σ, $-\sum_{\tau=1}^{t} Z_\tau^i$ and $-\bar{Z}_t^k(\sigma^{t-1})$ represent trader i's log-likelihood and sum of per period conditional expected log-likelihoods, respectively. Axioms A1 and A2 in B–E state standard assumptions on the payoff functions and the endowment stream. Axioms A3 and A5 are technical axioms introduced to avoid pathological behavior of log-likelihood ratios on the boundaries of the simplex.

BLUME–EASLEY THEOREM 8: *Assume Axioms* 1–3 *and* 5. *On the event*:

$$\lim_{t\to\infty}\left[t\log\frac{\beta_j}{\beta_i}+\bar{Z}_t^i(\sigma^{t-1})-\bar{Z}_t^j(\sigma^{t-1})\right]=+\infty, \tag{1}$$

$c^i(\sigma^t)\to 0$ *p-a.s.*

The key step in the proof of Theorem 8 is to show that p-a.s. the difference between the log-likelihood of two traders diverges if the difference between their sum of per period conditional expected log-likelihoods diverges: $\bar{Z}_t^i(\sigma^{t-1}) - \bar{Z}_t^j(\sigma^{t-1}) \uparrow \infty \Rightarrow \sum_{\tau=1}^{t}(Z_\tau^i - Z_\tau^j) \uparrow \infty$. To do so, B–E used the decomposition $\sum_{\tau=1}^{t}(Z_\tau^i - Z_\tau^j) = A_t^i\bar{Z}_t^i(\sigma^{t-1}) - A_t^j\bar{Z}_t^j(\sigma^{t-1})$, with $A_t^k(\sigma^t) = \frac{\sum_{\tau=1}^{t} Z_\tau^k}{\bar{Z}_t^k(\sigma^{t-1})}$, and argued that $A_t^k(\sigma^t)\to 1$ p-a.s. guarantees that $\bar{Z}_t^i(\sigma^{t-1}) - \bar{Z}_t^j(\sigma^{t-1}) \uparrow \infty \Rightarrow A_t^i\bar{Z}_t^i(\sigma^{t-1}) - A_t^j\bar{Z}_t^j(\sigma^{t-1}) \uparrow \infty$. This implication is not true. The reason is that A_t^i and A_t^j can converge to 1 at different rates.

Sandroni (2000) showed that the Strong Law of Large Numbers for martingale differences implies $\liminf \frac{1}{t}[\bar{Z}_t^i(\sigma^{t-1}) - \bar{Z}_t^j(\sigma^{t-1}] > 0 \Rightarrow \sum_{\tau=1}^{t}(Z_\tau^i - Z_\tau^j) \uparrow \infty$. Hence Theorem 8 can be fixed, replacing condition 1 with: $\log\frac{\beta_j}{\beta_i} + \liminf \frac{1}{t}[\bar{Z}_t^i(\sigma^{t-1}) - \bar{Z}_t^j(\sigma^{t-1})] > 0$.

2. THE COUNTEREXAMPLE

Consider an economy with two states $\mathcal{S} = \{a, b\}$ that occur with i.i.d. probability $p(a) = p(b) = \frac{1}{2}$. There are three traders $\{1, 2, 3\}$ with log utility (A1 is satisfied[3]), homogeneous discount factors, and positive endowment. The aggregate endowment is constant in each period (A2 is satisfied). Traders 1 and 2 have i.i.d. beliefs: $\forall t$, $p_t^1(a|\mathcal{F}_{t-1}) = p_t^2(b|\mathcal{F}_{t-1}) = \frac{1}{3}$.

Trader 3 is a Bayesian learner whose prior support consists of the two singletons $\{p_1, p_2\}$ to which he assigns equal probabilities. Therefore, in every period, $p_t^3(a|\mathcal{F}_{t-1}) \in [\frac{1}{3}, \frac{2}{3}]$ (A3–A5 are satisfied) and $\forall\sigma\in\Sigma, \forall t$, $p_t^3(\sigma) = \frac{1}{2}p_t^1(\sigma) + \frac{1}{2}p_t^2(\sigma)$.

The following claims show that Theorem 8 is false.

[3]If we define $\log 0 = -\infty$.

CLAIM 1: *Traders* 1 *and* 2 *satisfy Blume–Easley's sufficient condition to vanish. Let* $i = 1, 2$; *by construction*, $\beta_3 = \beta_i$, *hence* $t \log \frac{\beta_3}{\beta_i} = 0$. *The claim follows as* p-*a.s. we have that*:

$$\lim_{t\to\infty}\left[\bar{Z}_t^i(\sigma^{t-1}) - \bar{Z}_t^3(\sigma^{t-1})\right]$$
$$= \lim_{t\to\infty}\left[\sum_{\tau=1}^{t} E_p \log \frac{p_\tau(\sigma|\mathcal{F}_{t-1})}{p_\tau^i(\sigma|\mathcal{F}_{\tau-1})} - \sum_{\tau=1}^{t} E_p \log \frac{p_\tau(\sigma|\mathcal{F}_{\tau-1})}{p_\tau^3(\sigma|\mathcal{F}_{\tau-1})}\right] = \infty.$$

The first equality is by notation; *the second one follows because a divergent lower bound can be constructed using the following observations*:

(i) $\forall t \ \frac{1}{2}\log\frac{9}{8} = E_p \log \frac{p_t(\sigma|\mathcal{F}_{t-1})}{p_t^i(\sigma|\mathcal{F}_{t-1})} \geq E_p \log \frac{p_t(\sigma|\mathcal{F}_{t-1})}{p_t^3(\sigma|\mathcal{F}_{t-1})}$ *because* $p_t^3(\sigma|\mathcal{F}_{t-1}) \in \left[\frac{1}{3}, \frac{2}{3}\right] \forall t$.

(ii) *By symmetry*, $\sum_{\tau=1}^{t-1} I_\tau^a = \frac{t-1}{2} \Rightarrow p_t^3(\sigma|\mathcal{F}_{t-1}) = \frac{1}{2} \Rightarrow E_p \log \frac{p_t(\sigma|\mathcal{F}_{t-1})}{p_t^3(\sigma|\mathcal{F}_{k-1})} = 0$.

(iii) $\sum_{\tau=1}^{t-1} I_\tau^a = \frac{t-1}{2}$ *is a recurrent event*, *hence* $p\{\sum_{\tau=1}^{t-1} I_\tau^a = \frac{t-1}{2}\ \text{i.o.}\} = 1$.

CLAIM 2: *There is no path* σ *in which both Trader* 1 *and Trader* 2 *vanish. By contradiction, let* $\sigma \in \Sigma : c_t^1(\sigma) \to 0$ *and* $c_t^2(\sigma) \to 0$. *For* $j = 1, 2, 3$, *let* $\lambda_j \in (0, +\infty)$ *be the Pareto weights of the Social Planner Problem. The FOC for this economy would imply that* $\frac{\lambda_3 p_t^3(\sigma)}{\lambda_1 p_t^1(\sigma)} = \frac{c_t^3(\sigma)}{c_t^1(\sigma)} \to \infty$ *and* $\frac{\lambda_3 p_t^3(\sigma)}{\lambda_2 p_t^2(\sigma)} = \frac{c_t^3(\sigma)}{c_t^2(\sigma)} \to \infty$. *This is impossible. The left-hand side of at least one of the FOC must be less than* $+\infty$ *because, by construction*, $\forall \sigma \in \Sigma, \forall t$, $p_t^3(\sigma) = \frac{1}{2} p_t^1(\sigma) + \frac{1}{2} p_t^2(\sigma) \leq \max(p_t^1(\sigma), p_t^2(\sigma))$.

REFERENCES

BLUME, L., AND D. EASLEY (2006): "If You're so Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets," *Econometrica*, 74, 929–966. [849]

MASSARI, F. (2012): "Price Dynamics and Market Selection," Working Paper, Washington University in St. Louis. [849]

SANDRONI, A. (2000): "Do Markets Favor Agents Able to Make Accurate Predictions?" *Econometrica*, 68, 1303–1342. [849,850]

Dept. of Economics, Washington University in St. Louis, St. Louis, MO 63130, U.S.A.; fmassari@wustl.edu.

Manuscript received November, 2011; final revision received October, 2012.