A proof of Tanaka' s formula for Random Walk which does not rely on the discrete time version of Ito's lemma

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I present a simple proof of Tanaka's formula for Random Walk that does not rely on the discrete time version of Îto's lemma. Let $x_\tau = \begin{cases} 1 \; w.p \; \frac{1}{2} \\ -1 \; w.p \; \frac{1}{2} \end{cases}$, and $S_t = \sum_{\tau=1}^{t} x_\tau$.

Let $z_t = x_t\,(sign(S_{t-1}))$, with $sign(y) := \begin{cases} 1 \; if \; y \geq 0 \\ -1 \; if \; y < 0 \end{cases}$

The local time is defined as the number of periods in which $L_t \equiv \sum_{\tau=0}^{t} I_{S_\tau=0}$

Tanaka's formula for random walks 1. $\left|\sum_{\tau=1}^{t} x_\tau\right| = \sum_{\tau=1}^{t} z_\tau + L_t - 2I_{\{x_{T^*+1}=-1\}}$

Proof. For a generic path, define the following sequence of events:
$\Big\{T_1 := min_t\{t > 0 : \sum_{\tau=1}^{t} x_\tau = 0\}, ..., T_n := min_t\{t > T_{n-1} : \sum_{\tau=T_{n-1}+1}^{t} x_\tau = 0\}, ...\Big\}$. Each event

represents a period in which the RW is at 0. The following decomposition holds.

$$
\begin{aligned}
\left|\sum_{\tau=0}^{t} x_\tau\right| &= \left|\sum_{\tau=1}^{T^*=\max_n T_n<t} x_\tau + \sum_{\tau=T^*+1}^{t} x_\tau\right| \\
&=^a \ 0 + \left|\sum_{\tau=T^*+1}^{t} x_\tau\right| \\
&=^b \sum_{\tau=T^*+1}^{t} z_\tau - 2I_{\{x_{T^*+1}=-1\}} \\
&= \sum_{\tau=T^*+1}^{t} z_\tau - 2I_{\{x_{T^*+1}=-1\}} + \sum_{\tau=1}^{T^*} z_\tau - \sum_{\tau=1}^{T^*} z_\tau \\
&=^c \sum_{\tau=1}^{t} z_\tau + L_t - 2I_{\{x_{T^*+1}=-1\}}
\end{aligned}
$$

a: $\sum_{\tau=1}^{T^*=\max_n T_n<t} x_\tau = 0$ by definition in every T_n.

b: $\left|\sum_{\tau=T^*+1}^{t} x_\tau\right| = \sum_{\tau=T^*+1}^{t} z_\tau - 2I_{\{x_{T^*+1}=-1\}}$:

Proof.

- *i)* $\sum_{\tau=T^*+2}^{t} z_\tau = \sum_{\tau=T^*+2}^{t} x_\tau sign\left(S_{\tau-1}\right) = \left|\sum_{\tau=T^*+2}^{t} x_\tau\right|$.

  Because no crossing after T^* implies $\forall\tau \geq T^*+2, sign\left(S_{\tau-1}\right) = sign\left(x_{T^*+1}\right)$.

- *ii)* $x_{T^*+1} = +1 \Rightarrow \sum_{\tau=T^*+1}^{t} z_\tau = \left|\sum_{\tau=T^*+1}^{t} x_\tau\right|$

  Because, by the definition of $sign(.)$,

  $$x_{T^*+1} = +1 \Rightarrow x_{T^*+1} sign\left(S_{T^*}\right) + \sum_{\tau=T^*+2}^{t} x_\tau sign\left(S_{\tau-1}\right) = +1 + \left|\sum_{\tau=T^*+2}^{t} x_\tau\right| = \left|\sum_{\tau=T^*+1}^{t} x_\tau\right|$$

- *iii)* $x_{T^*+1} = -1 \Rightarrow \sum_{\tau=T^*+1}^{t} z_\tau = \left|\sum_{\tau=T^*+1}^{t} x_\tau\right| - 2$.

  Because, by the definition of $sign(.)$,

  $$x_{T^*+1} = -1 \Rightarrow x_{T^*+1} sign\left(S_{T^*}\right) + \sum_{\tau=T^*+2}^{t} x_\tau sign\left(S_{\tau-1}\right) = -1 + \left|\sum_{\tau=T^*+2}^{t} x_\tau\right| = \left|\sum_{\tau=T^*+1}^{t} x_\tau\right| - 2$$

□

c : We prove that $\sum_{\tau=1}^{T^*} z_\tau = -\frac{1}{2} 2L_t$

Proof. Define the two sets: $n^+ := \{n : x_{T_{(n-1)}+1} = +1\}$ and $n^- := \{n : x_{T_{(n-1)}+1} = -1\}$. n^+, n^- represents the number of periods that the random walk hits 0 from above and below, respectively.

$$n \in n^+ \Leftrightarrow x_{T_{(n-1)}+1} = +1 \Rightarrow^{By \quad ii)} \sum_{\tau=T_{(n-1)}+1}^{T_n} z_\tau = \left| \sum_{\tau=T_{(n-1)}+1}^{T_n} x_\tau \right| = 0$$

$$n \in n^- \Leftrightarrow x_{T_{(n-1)}+1} = -1 \Rightarrow^{By \quad iii)} \sum_{\tau=T_{(n-1)}+1}^{T_n} z_\tau = \left| \sum_{\tau=T_{(n-1)}+1}^{T_n} x_\tau \right| - 2 = 0 - 2$$

Clearly, $n = n^+ + n^-$.

The result follows because: $\sum_{\tau=1}^{T^*} z_\tau = \sum_{n^+} \sum_{\tau=T_{(n-1)}+1}^{T_n} z_\tau + \sum_{n^-} \sum_{\tau=T_{(n-1)}+1}^{T_n} z_\tau$ and noticing that, by the Law of Large Number, we expect the RW to be positive half of the time: $\frac{n^+}{n} \to \frac{1}{2}$. □

□