The Wisdom of the Crowd in Dynamic Economies*

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Abstract

The Wisdom of the Crowd applied to financial markets asserts that prices, an average of agents' beliefs, are more accurate than individual beliefs. However, a market selection argument implies that prices eventually reflect only the beliefs of the most accurate agent. In this paper, we show how to reconcile these alternative points of view. In markets in which agents naively learn from equilibrium prices, a dynamic Wisdom of the Crowd holds. Market participation increases agents' accuracy, and equilibrium prices are more accurate than the most accurate agent. If we replace naive learning with Bayes' rule, this positive result disappears.

Keywords: Wisdom of the Crowd, Heterogeneous Beliefs, Market Selection Hypothesis, Naive Learning.
JEL Classification: D53, D01, G1

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1 Introduction

The informational content of prices is a central issue in the analysis of equilibria of competitive markets. In financial markets, in particular, asset prices are often believed to be good predictors of the economic performance of the underlying fundamentals. Three different mechanisms have been proposed as possible explanations for this remarkable property. The rational expectation and the learning-from-price literatures argue that equilibrium prices are accurate because they reveal and aggregate the information of all market participants. The Market Selection Hypothesis, *MSH*, proposes instead that prices become accurate because they eventually reflect only the beliefs of the most accurate agent. The Wisdom of the Crowd argument, *WOC*, however suggests that market prices are accurate because individual, idiosyncratic errors are averaged out by the price formation mechanism.

Although these theories aim to explain the same phenomenon, they rest on different and somehow conflicting hypotheses. In the learning-from-price literature, all agents are assumed to agree on the way to interpret information. In equilibrium, when all private information gets revealed, all agents must hold the same belief because they cannot "agree to disagree." Therefore, the MSH and the WOC arguments are void. By contrast, in the MSH and WOC literatures, agents can disagree on how to interpret information about fundamentals. However, existing models of market selection are incompatible with WOC because they do not allow for belief heterogeneity in the long-run: by selecting the most accurate agent, the market destroys all accuracy gains that could be achieved by balancing out agents' opposite biases.

To reconcile these three mechanisms, we propose a market selection model in which agents believe that asset prices are a good predictor of the fundamentals, but lack what is needed to extract information from them. We assume that each agent's beliefs for next-period states are formed by giving weight to two different models. The first model, *market probabilities*, is common to all agents and coincides with the prediction implied by the market consensus. The second model, *dogmatic probabilities*, is agent specific and represents everything that each agent has learned according to his subjective probabilistic view of the world. This rule of thumb, first introduced by Manski (2006) in the context of static prediction markets, captures the idea that agents' opinions might depend on equilibrium prices in a way that is not fully rational. Full rationality would require each agent to have a correct model of how other agents process information — a situation which never occurs in practice. On the contrary, agents in our model settle on a second best: they naively incorporate other agents' opinions by anchoring their beliefs to equilibrium prices. We interpret this rule as depicting the beliefs of an agent who tries to find a compromise between his subjective view about fundamentals and the possibility that markets might be accurate after all.

In this paper, we describe how this simple rule affects the equilibrium dynamics of agents' consumption-shares and asset prices in an otherwise standard dynamic stochastic general equilibrium model with complete markets (as in Sandroni, 2000; Blume and Easley, 2006). The following are our main findings.

First, we define WOC as the situation in which price probabilities are more accurate than all dogmatic probabilities, and discuss the conditions under which it occurs. Under mild assumptions (i.e., there are at least two agents whose dogmatic probabilities are biased in different directions), we show that a WOC emerges when agents sufficiently weigh the market probability in forming their beliefs. In this case, the equilibrium path exhibits long-run heterogeneity, market probabilities never settle down, and WOC occurs.

Second, we show that market accuracy is a virtuous self-fulfilling prophecy. All else equal, if market participants believe that prices are accurate, prices are indeed accurate. In markets populated by agents whose beliefs follow our rule, selection forces determine a market probability process that is mean-reverting around the value that makes all surviving agents equally accurate. We prove our result by approaching the limiting case in which agents put all the weight on market probabilities when forming their predictions. Prices become accurate because giving more weight to market probabilities makes agent beliefs more similar, thus reducing the volatility of the market probability process and pushing the turning value of the (mean-reverting) price process closer to the truth.

Third, we identify and characterize the key component of agent beliefs for survival, and thus also for the WOC to hold. We show that survival does not only depend on the accuracy of agents' dogmatic probabilities, but also on agent contribution to the diversity of the market opinion. If the dogmatic probabilities of two agents are biased in the same way, only the most accurate agent survives because the beliefs of the least accurate do not contribute to market diversity. However, if the dogmatic probabilities of two agents are not biased in the same way, both agents survive, and the WOC occurs. Furthermore, we prove that the belief formation rule we consider provides an evolutionary advantage to agents. The survival chances of an agent are (weakly) higher if his beliefs incorporate market probabilities rather than blindly trust his dogmatic opinion.

Lastly, we add a layer of sophistication to our belief formation rule and discuss the case in which agents learn in a Bayesian way how much weight to give to the market probability component of their beliefs. This layer of sophistication guaranties agents survival but destroys the WOC. In economies in which all agents use Bayes' rule to update the relative weight they attach to market probabilities, the WOC does not occur. Instead, all agents survive and coordinate their beliefs on the most accurate dogmatic probabilities.

The paper proceeds as follows. In the next section, we discuss the related literature. In Section 2, we introduce the model and characterize agent beliefs. Section 3 defines *accuracy* and *WOC*. The main results are presented in Section 4: first, we provide sufficient conditions for a dynamic WOC to occur; then, we prove that market accuracy is a virtuous self-fulfilling prophecy. Section 5 discusses agents survival and the relation between learning and WOC. Throughout the paper we use simulations for illustrative purposes; their length varies to accommodate the different convergence rates. Proofs are in two Appendices.

1.1 Related literature

A very influential stream of literature argues that asset prices are accurate because financial markets are an efficient aggregator of private information (Grossman, 1976, 1978; Radner, 1979; Grossman and Stiglitz, 1980). Closely related to the literature on information transmission (Aumann, 1976; Geanakoplos and Polemarchakis, 1982), this literature assumes that agents disagree due solely to differences in their private information and provides conditions under which the price formation mechanism reveals all private information to all agents in the market. Because all agents agree on the way to interpret information, and prices instantaneously reveal all available information, in equilibrium all agents must hold the same beliefs and no WOC or selection based on belief heterogeneity can occur. Prices are accurate because they reflect and aggregate all relevant information. However, it is hard to imagine that most agents active in financial markets can agree on what information is relevant and how to interpret it — *"Ordinary investors have no model or at best a very incomplete model of the behavior of prices, dividends, or earnings of speculative assets" — Shiller (1984).* In fact, there is overwhelming evidence documenting the inability of agents to process information "rationally," even in simple experimental settings (Kahneman, 2011), and that agents who use well established models might be acting irrationally by failing to account for transaction costs (Barber and Odean, 1999) or estimation errors (DeMiguel et al., 2009).

An alternative explanation for market accuracy, the MSH, relies on the evolutionary argument that markets become accurate because they select for accurate agents (Alchian, 1950; Friedman, 1953). According to the MSH, agents with inaccurate beliefs progressively lose their wealth to accurate agents. By standard economic arguments, equilibrium prices are asymptotically accurate because they reflect only the beliefs of the most accurate agent in the economy (Sandroni, 2000). The MSH setting allows agents to disagree on the way to interpret information. However, the selection result is far from encouraging. By selecting for a unique most accurate agent, the market "destroys" all the accuracy gains that could be achieved by pooling the diverse opinions of the agents who vanish. The market does not work as an aggregator, and no WOC can occur. Market prices can only be as accurate as the most accurate agent (Blume and Easley, 2009), even in the knife-edge cases in which there are multiple survivors (Jouini and Napp, 2011; Massari, 2013). In addition to our model, others in the market selection literature allow for long-run survival of agents with heterogeneous beliefs. Survival of agents with heterogeneous beliefs occurs in economies with incomplete markets (Beker and Chattopadhyay, 2010; Cogley et al., 2013; Cao, 2017), ambiguous averse agents (Guerdjikova and Sciubba, 2015), exogenous saving rules (Bottazzi and Dindo, 2014; Bottazzi et al., 2017), and recursive preferences (Borovička, 2015; Dindo, 2015). Unlike ours, however, these models do not deliver WOC because there is no feedback between agent beliefs and equilibrium prices. A model that merges elements of rational learning from prices and selection is Mailath and Sandroni (2003). This model does not endogenously generate WOC because long-run heterogeneity is a consequence of the presence of noise traders.

Finally, the WOC argument (initially proposed by Galton, 1907, and recently popularized by Surowiecki, 2005), hypothesizes that asset prices are accurate because the opposite, idiosyncratic errors of individual agents are averaged out by the price formation mechanism. The WOC hypothesis has inspired a growing interest in prediction markets (Wolfers and Zitzewitz, 2004; Arrow et al., 2008) and social trading platforms (Chen et al., 2014; Pelster et al., 2017). Within the prediction markets literature, most of the attention has been focused on static settings. However, there is no solid foundation to justify the WOC argument. WOC can occur only if the consumption-shares/beliefs distribution is such that individual mistakes cancel out. The main limitation of WOC is the lack of theoretical arguments supporting that this is indeed the case. Further, there is evidence that even if agents were rationally processing private unbiased signals, the aggregate beliefs might be biased nevertheless (Ali, 1977; Manski, 2006; Ottaviani and Sørensen, 2014). Works that also combine dynamic elements such as ours in prediction markets are Kets et al. (2014) and Bottazzi and Giachini (2016). The WOC has also been investigated within other contexts. In the literature of social learning in networks, Golub and Jackson (2010) and Jadbabaie et al. (2012) provide conditions under which agents imitating each other and naively updating their beliefs — using a rule similar to ours — can achieve the same outcome of rational learning models. In the literature on collective problem-solving, Hong and Page (2004) explore the trade-off between opinion diversity and the difficulty in identifying optimal solutions (see also Page, 2007).

Our model brings together the contributions of these three branches of literature. Agents incorporate price in their beliefs, but not in a fully rational way; the market selects against traders whose opinions are inaccurate, but only if those trader beliefs cannot be used to increase market accuracy; and a dynamic WOC emerges: the market endogenously determines consumption shares that make the average beliefs more accurate than that of the most accurate agent in isolation.

2 The model

We study a standard dynamic stochastic exchange economy with complete markets where agents have heterogeneous beliefs on the realizations of states of nature. Assuming complete markets implies that agents can use contracts to exchange contingent commodities for any date and any state. Since agents have heterogeneous beliefs but are otherwise identical, they assign different evaluations for contingent commodities and use the available assets to trade on such differences. Market clearing determines equilibrium prices and allocations. At equilibrium, the agent who assigns a higher probability to a certain event takes a long position (in excess of his equilibrium consumption if beliefs were homogeneous) in the asset paying a unit of the consumption good in that event. The agent with a lower probability supplies the asset. We are interested in studying the resulting consumption-share and asset-price dynamics and in characterizing their long-run properties. The central question is if market probabilities, a proper normalization of asset prices, become accurate.

Time is discrete, indexed by t, and begins at date $t = 0$. In each period $t \geq 1$, the economy can be in one of S mutually exclusive states, $\mathcal{S}$. The set of partial histories until t is the Cartesian product $\Sigma^t = \times^t \mathcal{S}$ and the set of all paths is $\Sigma := \times^\infty \mathcal{S}$. $\sigma = (\sigma_1, ...)$ is a representative path, $\sigma^t = (\sigma_1, ..., \sigma_t)$ is a partial history until period t, and $\mathcal{F}_t$ is the σ-algebra generated by the cylinders with base σ^t. By construction $\{\mathcal{F}_t\}$ is a filtration and $\mathcal{F}$ is the σ-algebra generated by their union. For any probability measure ρ on $(\Sigma, \mathcal{F})$, $\rho(\sigma^t) := \rho(\{\sigma_1 \times ... \times \sigma_t \times S \times S \times ...\})$ is the marginal probability of the partial history σ^t while $\rho(\sigma_t|\sigma^{t-1}) = \frac{\rho(\sigma^t)}{\rho(\sigma^{t-1})}$ is the conditional probability of σ_t given σ^{t-1}. ρ_t is the measure on $(\mathcal{S}, 2^{\mathcal{S}})$ defined by $\rho_t := \rho(\cdot|\sigma^{t-1})$. P denotes the true probability measure on $(\Sigma, \mathcal{F})$. We shall assume that states of nature are i.i.d. with $P_t = P$ for all $t \geq 1$ for a measure P

on $(\mathcal{S}, 2^{\mathcal{S}})$.[1]

Next, we introduce a number of economic variables with time index t. All these variables are adapted to the information filtration $\{\mathcal{F}_t\}$.

The economy contains I agents, $\mathcal{I} = \{1, \ldots, I\}$. For all paths σ, each agent $i \in \mathcal{I}$ is endowed with a stream of the consumption good, $(e^i_t(\sigma))_{t=0}^{\infty}$. We take the consumption good in $t = 0$ as the numeràire of the economy. Each agent's objective is to maximize the stream of discounted expected utility he gets from consumption. Expectations are computed according to agent beliefs p^i, a measure on $(\Sigma, \mathcal{F})$. Beliefs are heterogeneous and agents agree to disagree. Naming $q(\sigma^t)$ the date $t = 0$ price of the asset that delivers one unit of consumption in event σ^t and none otherwise, agent i maximization reads:

$$\max_{(c^i_t(\sigma))_{t=0}^{\infty}} \mathrm{E}_{p^i}\left[\sum_{t=0}^{\infty} \beta^t u^i(c^i_t(\sigma))\right] \qquad s.t. \quad \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)\left(c^i_t(\sigma) - e^i_t(\sigma)\right) \leq 0.$$

A competitive equilibrium is a sequence of prices and, for each agent, a consumption plan that is preference maximal on the budget set, and such that markets clear in every period: $\forall(\sigma, t), \sum_{i\in\mathcal{I}} e^i_t(\sigma) = \sum_{i\in\mathcal{I}} c^i_t(\sigma)$. Assumptions **A1-A3** below are taken to ensure that a unique competitive equilibrium exists (Peleg and Yaari, 1970) and that state prices aggregate the different views of the agents without distortions (Rubinstein, 1974; Blume and Easley, 1993).[2]

A1 All agents have the same discount factor β, evaluate consumption using a log utility, and can exchange a complete set of contracts.

A2 The aggregate endowment is constant: $\sum_{i\in\mathcal{I}} e^i_t(\sigma) = 1$ for all t and σ.

[1] With an abuse of notation, we are denoting with P both a measure on states and a measure on sequences, depending on the context.

[2] Our analysis can be generalized to economies with non-log preferences and aggregate risk. However, the mapping between beliefs and prices would be less transparent (Dindo, 2015; Massari, 2017).

A3 For all agents $i \in \mathcal{I}$, all dates t and all events σ^t, $p^i(\sigma^t) > 0 \Leftrightarrow P(\sigma^t) > 0$.

As it is customary in the market selection literature, the asymptotic fate of an agent is characterized by his consumption-shares as follows.

Definition 2.1. *Agent i* vanishes *if* $\lim_{t\to\infty} c_t^i(\sigma) = 0$ *P-a.s., he* survives *if* $\limsup_{t\to\infty} c_t^i(\sigma) > 0$ *P-a.s., he* dominates *if* $\lim_{t\to\infty} c_t^i(\sigma) = 1$ *P-a.s..*

2.1 Agent beliefs

We assume that each agent's beliefs for next-period states are formed by giving weight to two different models. The first model, *market probabilities*, is common to all agents and coincides with the prediction implied by the market consensus. The second model, *dogmatic probabilities*, is agent specific. For simplicity, we assume that dogmatic probabilities π^i are constant and that agents agree on the fact that all states are possible (strict positivity).[3] A natural choice for *market probabilities* is the risk-neutral probabilities of the states because a representative agent exists under **A1-A3**, and the risk-neutral probability of the states reflects his unbiased beliefs (Rubinstein, 1974; Blume and Easley, 1993). The next lemma reminds the reader of the analytic form of market probabilities.

Lemma 2.1. *(Rubinstein, 1974). Under **A1-A3**, market probabilities are:*

$$\forall(t,\sigma), \quad \begin{cases} p^M(\sigma^t) = \sum\limits_{i\in\mathcal{I}} p^i(\sigma^t) c_0^i \\ p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) c_{t-1}^i(\sigma) \end{cases} . \tag{2.1}$$

We are now ready to define the beliefs of the agents in our economy.

[3] Most of our proofs generalize verbatim to the non-i.i.d. setting with minor notational changes at the expense of less intuitive definitions. The strict positivity requirement for all measures can also be relaxed.

Definition 2.2. *For all $i \in \mathcal{I}$, agent i beliefs are given by*

$$\forall (t,\sigma), \quad \begin{cases} p^i(\sigma^t) = \prod_{\tau \le t} p^i(\sigma_\tau|\sigma^{\tau-1}) \\ p^i(\sigma_t|\sigma^{t-1}) = (1-\alpha^i)p^M(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t) \end{cases} ; \tag{2.2}$$

where π^i is a strictly positive measure on $(\mathcal{S}, 2^{\mathcal{S}})$ and $\alpha^i \in (0,1]$.[4]

This rule describes the attitude of an agent who partially believes that markets are accurate. The parameter α^i determines how much agent i believes in market accuracy. Having $\alpha^i = 1$ represents the extreme scenario in which agent i ignores the market. This is the standard case in the market selection literature, where most of the models make the simplifying assumptions that agent beliefs are independent of each other and of equilibrium prices. Whereas $\alpha^i = 0$ represents the case in which agent i gives no weight to his dogmatic probabilities because he is certain that markets are accurate — with a similar attitude to the economist who finds a \$20 bill lying on the ground and refuses to believe it. The intermediate cases of $\alpha^i \in (0,1)$ are consistent with the attitude of an agent who biases his opinion in the direction of the market consensus. This is a mental attitude that is consistent with many known biases including anchoring (Shiller, 1999) and herding (Lakonishok et al., 1992).

In the context of static prediction markets, rule 2.2 has been used to discuss the effect of agents' partial learning from equilibrium prices (Manski, 2006). In the learning literature on networks, a similar rule is used by Jadbabaie et al. (2012), while in the portfolio theory literature, beliefs (2.2) determine a portfolio that coincides with the Fractional-Kelly rule (MacLean et al., 2011; Kets et al., 2014), a mixture between the Kelly portfolio (derived according to his dogmatic probabilities) and the market portfolio. The Fractional-Kelly rule is often recommended

[4] We rule out $\alpha^i = 0$ because $\alpha^i = 0$ for all $i \in \mathcal{I}$ leads to an indeterminate equilibrium.

because it reduces the volatility of the Kelly rule and yet ensures a high growth rate of capital. Moreover, its hedging properties meet the reputation incentives of the professional investors: *Worldly wisdom teaches that it is better for reputation to fail conventionally than to succeed unconventionally (Keynes, 1936).*

We conclude this section by reassuring the reader that if agent beliefs satisfy Definition 2.2, then the competitive equilibrium exists and is unique — because **A3** is satisfied. Moreover, on every equilibrium path, both market probabilities and agent beliefs belong to the convex combination of agents dogmatic probabilities (see Lemma A.1 in Appendix A).

3 Agents accuracy and survival

In this section, we formally define accuracy and discuss how it affects agents survival. Following an established tradition in the market selection literature (Blume and Easley, 1992), we use the *average* (conditional) *relative entropies* (Kullback-Leibler divergences), to rank agents' accuracy.

Definition 3.1. *The* average relative entropy *between a probability p^i and the true probability P is*

$$\bar{d}(P||p^i) := \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} d(P||p^i_\tau),^{5}$$

where, for all τ, $d(P||p^i_\tau) := \mathrm{E}_P \left[\ln \frac{P(\sigma_\tau)}{p^i(\sigma_\tau|\sigma^{\tau-1})}\right]$.

Definition 3.2. *Agent i is* more accurate *than agent j if* $\bar{d}(P||p^i) < \bar{d}(P||p^j),\ P$*-a.s.*. *Agent i is* as accurate as *agent j if* $\bar{d}(P||p^i) = \bar{d}(P||p^j),\ P$*-a.s.*.

[5] Lemma 6.1.2, pg.133, of Gray (2010) guarantees that this limit exists P-a.s. — ergodic property. The *average relative entropy* has the ergodic property because (i) i.i.d. states of nature imply that all indicator functions on $(\Sigma, \mathcal{F})$ have the ergodic property and (ii) the conditional relative entropy of market probabilities, and thus also of agent beliefs, is a bounded measurement on $(\Sigma, \mathcal{F})$.

The average relative entropy is uniquely minimized at $p^i = P$, strictly convex, and $d(P||\pi) = \bar{d}(P||\pi)$ whenever P and π are constant over time. This notion of accuracy is commonly adopted in the selection literature because of its straightforward implications on agents survival. Under reasonable assumptions, the simple comparison of agents' relative entropy delivers a sufficient condition for an agent to vanish.

Proposition 3.1. *(Sandroni, 2000). Under **A1-A3**, agent i vanishes if there exists an agent $j \in \mathfrak{I}$ who is more accurate:*

$$\bar{d}(P||p^j) < \bar{d}(P||p^i) \ P\text{-a.s.} \Rightarrow \text{Agent } i \text{ vanishes.}$$

An alternative approach to studying agents survival is to directly compare agents' relative entropy against that of market probabilities (Massari, 2017). This general approach can be used to deliver a sufficient condition for an agent to vanish that is directly informative about the accuracy of market probabilities.[6]

Proposition 3.2. *Under **A1-A3**,*

(*a*) *no agent can be more accurate than the market:*

$$\forall i \in \mathfrak{I}, \bar{d}(P||p^i) \geq \bar{d}(P||p^M) \ P\text{-a.s.};$$

(*b*) *agent i survives only if he is as accurate as the market:*

$$\text{Agent } i \text{ survives} \Rightarrow \bar{d}(P||p^i) = \bar{d}(P||p^M) \ P\text{-a.s.}.$$

Proof. See Appendix A. □

[6] Massari (2017)'s condition for an agent to vanish is both necessary and sufficient. The necessary part of Massari (2017)'s condition is lost here because we are using relative entropy comparisons rather than likelihood ratios.

Proposition 3.2 significantly simplifies our analysis because standard techniques to approximate market probabilities and agents' relative entropies cannot be used when agent beliefs endogenously depend on equilibrium prices. Definition 2.2 generates beliefs that are path-dependent and not exchangeable — that is, no statistics of the data is informative enough to characterize the dynamics of our system. Most of our results are obtained by combining Propositions 3.1 and 3.2, and by taking advantage of the convexity of the relative entropy.

3.1 A definition of Wisdom of the Crowd

We say that WOC occurs if market probabilities are more accurate than the beliefs of the most accurate agent in isolation. Two quantities play a special role in our definition: the Best Individual Probability (π^{BIP}), which is the most accurate dogmatic probability, and the Best Collective Probability (π^{BCP}), which is the most accurate combination of agents' dogmatic probabilities. Moreover, we say that dogmatic probabilities are diverse when the Best Collective Probability differs from the Best Individual Probability — that is, if it is possible to combine dogmatic probabilities into a prediction that is more accurate than that of all dogmatic probabilities.

Definition 3.3. *Given a set of dogmatic probabilities* $\{\pi^1, ...\pi^I\}$*:*

- *the* Best Individual Probabilitys *is:* $\pi^{BIP} = \underset{p \in \{\pi^1,...,\pi^I\}}{\operatorname{argmin}} \bar{d}(P||p)$*;*
- *the* Best Collective Probabilitys *is:* $\pi^{BCP} = \underset{p \in Conv(\pi^1,...,\pi^I)}{\operatorname{argmin}} \bar{d}(P||p)$*;*
- *dogmatic probabilities are* diverse *when* $\pi^{BCP} \neq \pi^{BIP}$ *.*

WOC occurs if market probabilities are more accurate than the beliefs of the most accurate agent in isolation. By Lemma 2.1 and beliefs (2.2), when an agent is alone in the economy, both his beliefs and market probabilities coincide with his dogmatic probabilities. Thus, we can say that WOC occurs when market probabilities are more accurate than π^{BIP}.

Definition 3.4. *Under **A1-A3**, we say that* WOC occurs *if market probabilities are more accurate than the most accurate dogmatic probability:*

$$\bar{d}(P||p^M) < \bar{d}(P||\pi^{BIP}) \ P\text{-}a.s..$$

To gain intuition, consider a two-state, $\mathcal{S} = \{u, d\}$, two-agent, $\mathcal{I} = \{1, 2\}$, economy. The true probability of state u is, $P(u) = .5$. Agent 1 is pessimistic about u, his dogmatic probability is $\pi^1(u) = .4 < P(u)$; while agent 2 is optimistic, his dogmatic probability is $\pi^2(u) = .7 > P(u)$. Clearly, agent 1 has the most accurate dogmatic probabilities, thus $\pi^{BIP} = \pi^1 = .4$; while the most accurate way to combine the dogmatic probabilities of the two agents is $\pi^{BCP} = \frac{1}{2} = P$. WOC occurs if market probabilities are more accurate than the dogmatic probability of agent 1 (and thus 2) — in other words, if the market consensus is more accurate than all market participants in isolation.

The following Proposition shows that, irrespective of the mixing coefficients of agents, market probabilities are at least as accurate as π^{BIP} and at most as accurate as π^{BCP}. If WOC does not occur, selection forces ensure that market probabilities are as accurate as π^{BIP}. Otherwise, π^{BCP} represents an upper bound on market accuracy.

Proposition 3.3. *Under **A1-A3**, for every choice of $\alpha^i \in (0, 1]$ for all $i \in \mathcal{I}$, the market is at least as accurate as π^{BIP} and at most as accurate as π^{BCP}:*

$$\bar{d}(P||\pi^{BCP}) \leq \bar{d}(P||p^M) \leq \bar{d}(P||\pi^{BIP}), \ P\text{-}a.s..$$

Proof. See Appendix A. □

Proposition 3.3 is proven showing that in the long-run either the agent with the most accurate dogmatic probabilities dominates, and market probabilities are as accurate as π^{BIP}, or there is long-run heterogeneity, and market probabilities are a convex combination of the surviving agents' dogmatic probabilities — thus, by Definition 3.3, at most as accurate as π^{BCP}.

4 Main results

4.1 Necessary conditions for WOC

In this section, we establish two necessary conditions for WOC.

Proposition 4.1. *Under **A1-A3**, WOC can occur only if agent beliefs depend on prices.*

Proof. See Appendix A. □

Most of the results in the market selection literature do not generate WOC because they assume that agent beliefs are independent of each other and equilibrium prices. Under this assumption, the selection result is only partially positive because it creates a tension between selection and long-run survival of agents with heterogeneous beliefs. If one of the agents knows the true probability, market probabilities converge to his beliefs and become accurate. Otherwise, the market, by selecting the most accurate agent, destroys all potential accuracy gains that could be achieved by mixing the incorrect beliefs of agents with opposite biases (Jouini and Napp, 2011; Massari, 2017).

For example, suppose the market has an optimistic and a pessimistic agent. If the pessimistic agent is less accurate than the optimist, then the pessimist vanishes, and market probabilities become optimistic. Clearly, this is not the best way to make use of agent opinions. A better way would be to redistribute consumption-shares in such a way that market probabilities become accurate by balancing the opposite biases of the two agents. However, this is impossible when agents' beliefs are independent of each other because only the most accurate trader survives (Blume and Easley, 2009).

Proposition 4.2. *Under **A1-A3**, WOC can occur only if dogmatic probabilities are diverse.*

Proof. See Appendix A. □

This result implies that no WOC can occur if there is an agent whose belief accuracy cannot be improved by mixing his dogmatic probabilities with those of other agents. For example, in an economy with two states in which all dogmatic probabilities are biased in the same way, no WOC can occur because the most accurate combination of agent beliefs, π^{BCP}, is the one obtained by giving all wealth to the least biased among the agents (BIP). Another implication of Proposition 4.2 is that no WOC can occur if there is an agent whose dogmatic probabilities coincide with the truth. In this case, $\pi^{BIP} = P$ and mixing agent BIP beliefs with those of the others can only compromise his accuracy.

4.2 Sufficient conditions for WOC

The WOC occurs if the following conditions are simultaneously satisfied. First, it must be possible to achieve accuracy gains by balancing the different opinions of market participants — in other words, dogmatic probabilities must be diverse. Second, agents must believe strongly enough in market accuracy — α^i must be small for all agents $i \in \mathcal{I}$. Under these conditions, selection forces induce a non-degenerate consumption-share distribution, which guarantees that market probabilities are more accurate than the most accurate agent in isolation.

Proposition 4.3. *Under **A1-A3**, provided that dogmatic probabilities are diverse, there exists an $\bar{\alpha} \in (0,1)$ such that if $\alpha^i < \bar{\alpha}$ for all $i \in \mathcal{I}$, then WOC occurs and at least two agents survive.*

Proof. See Appendix A. □

For intuition, consider again an economy with two states, $\mathcal{S} = \{u, d\}$, and two agents $\mathcal{I} = \{1, 2\}$. The true probability of state u is $P(u) = .5$. Agent 1 is pessimistic about u, while agent 2 is optimistic. Their dogmatic probabilities are $\pi^1(u) = .4$ and $\pi^2(u) = .7$, respectively. As we noted earlier $\pi^{BIP}(u) = .4 \neq .5 = \pi^{BCP}(u) = P(u)$. Thus, agent beliefs are diverse and it is possible to achieve accuracy gains by mixing their opinions.

Figure 1 [top] shows that long-run heterogeneity and WOC occurs if agents give enough weight to market probabilities. With $\alpha^1 = \alpha^2 = .2$, we have long-run heterogeneity and WOC because the dependency of trader beliefs on market probabilities makes it impossible for any trader to dominate. When agent 1 (2) consumption-shares become large, his dogmatic probabilities have a large impact on market probabilities, making his beliefs less accurate than those of agent 2 (1). Thus, consumption-shares never find a resting point, market probabilities remain close to P and are more accurate than π^{BIP}. Formally, the consumption-shares are mean-reverting processes around the value $\bar{c}^1$ that determines a market probability $\bar{p}^M$ which makes agents 1 and 2 equally accurate (i.e. $c_t^1 \gtreqless \bar{c}^1 \Leftrightarrow d(P||p_t^1) \gtreqless d(P||p_t^2)$). WOC occurs because $\bar{p}^M$ is more accurate than π^1 and π^2, and market probabilities stay close to $\bar{p}^M$ a large enough number of periods.

Conversely, Figure 1 [bottom] shows that if agents do not give enough weight to market probabilities WOC does not occur because only one agent survives. With $\alpha^2 = .9$, agent 2 vanishes because he is less accurate than agent 1 for every consumption-share distribution: $\forall c_t^1, d(P||p_t^2) > d(P||\pi_t^1)$. This can be verified by noticing that agent 2's beliefs are less accurate than agent 1's even when agent

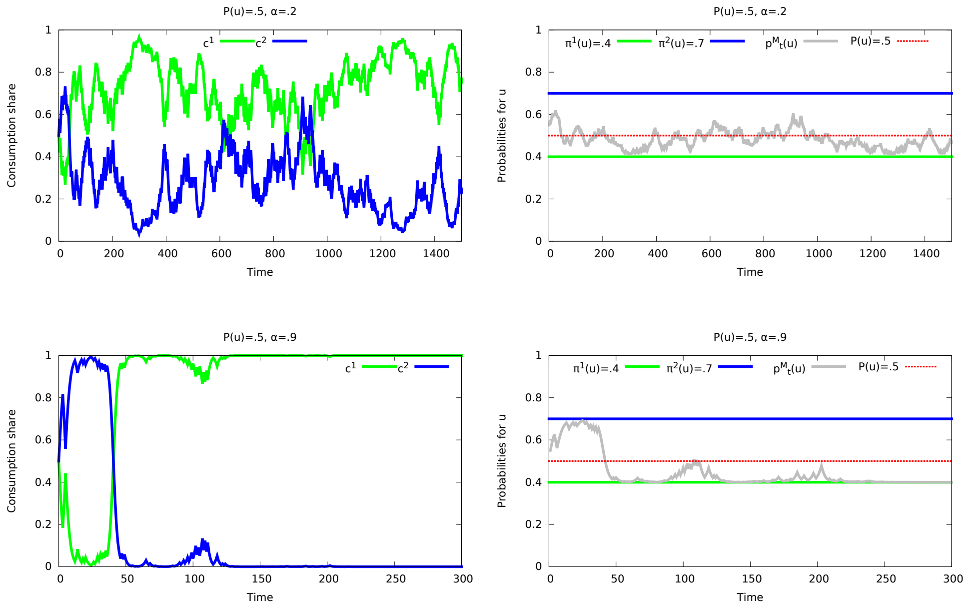

Figure 1: Consumption-shares [left] and market probability [right] dynamics in two economies with identical, diverse dogmatic probabilities, $P(u) = \pi^{BCP}(u) = .5 \neq \pi^{BIP}(u) = .4.$ and different mixing coefficients. [Top]: agents believe enough in market accuracy, $\alpha^1 = \alpha^2 = .2$, and WOC occurs. Consumption-shares never find a resting point, and market probabilities are more accurate than π^{BIP}. [Bottom]: agents do not believe enough in market accuracy, α^1=α^2=.9, and no WOC occurs. Agent 1 dominates, and market probabilities are as accurate as his dogmatic probabilities.

1 dominates and sets equilibrium prices equal to his dogmatic probabilities π^1: $p^2|_{p^M=\pi^1} = .1(.4) + .9(.7) = .67 \Rightarrow d(P||p^2|_{p^M=\pi^1}) > d(P||\pi^1)$.

4.3 Accurate markets: A self-fulfilling prophecy

Here we demonstrate that if agents in the economy are (almost) sure that markets are accurate, then markets are indeed (almost) accurate. By strongly relying on market probabilities, agents generate a virtuous interaction that makes both their beliefs and the market more accurate: selection forces endogenously find the best way to aggregate the diverse opinions of agents.

Theorem 4.1. *Let* $(\mathcal{E}_\alpha)$ *be a family of economies that satisfies* ***A1-A3*** *with* $\alpha^i = \alpha$ *for all* $i \in \mathcal{I}$ *and name each economy market probabilities process* (p^M_α)*. All economies have two states*[7] *and are identical in all respects except the value of agent mixing coefficients,* α*. The following result holds* P*-a.s.:*

$$\lim_{\alpha \to 0} \bar{d}(P||p^M_\alpha) = \bar{d}(P||\pi^{BCP}).$$

Proof. See Appendix B. □

Theorem 4.1 is proven by showing that, (i) a lower α implies that the point $\bar{p}^M$ where the market belief process reverts its drift is closer to π^{BCP}. And (ii), for every interval around $\bar{p}^M$, α can be chosen small enough to ensure that the market belief process spends most of its period in that interval. The difficulty in proving the result is that a lower α implies a lower variance, but also a weaker mean-reverting drift of the market probability process — the selection forces are weaker because agent beliefs become more similar. Thus, we have to determine which effect dominates when α is small. To make things worse, the per-period variances and drifts changes over time and are path-dependent. We prove that the accuracy gain for a more accurate mean-reverting point and a lower variance of the market probability process more than compensates for the accuracy loss due to weaker mean-reverting forces. Although market probabilities take longer

[7] For tractability reasons, we restrict our analysis to two-state economies. Our proof suggests that this result should hold also in economies with more than two states.

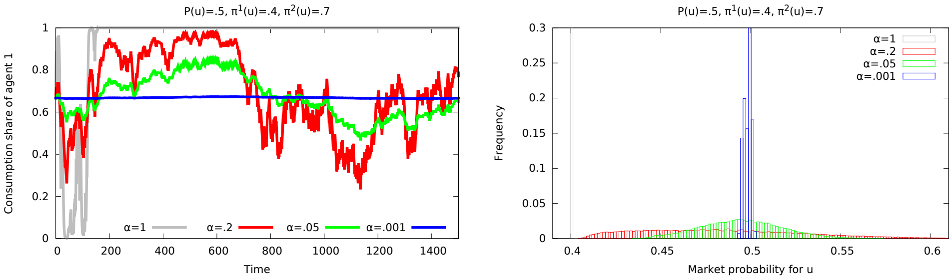

Figure 2: Consumption-share dynamics [left] and market belief frequencies [right] in four economies with true probability $P(u) = .5$, two agents with dogmatic probabilities $\pi^1(u) = .4$ and $\pi^2(u) = .7$, and four different values of α. The figure shows that a smaller α determines market belief frequencies that are more concentrated around the truth. Market beliefs empirical frequencies

to reach π^{BCP}, they are nevertheless more accurate because they remain closer to π^{BCP} after they reach it.

Figure 2 illustrates the result by showing the consumption-share dynamics and the frequency of market probabilities of four economies that only differ in their value of α. All economies have two agents with dogmatic probabilities $\pi^1(u) = .4$ and $\pi^2(u) = .7$, so that $\pi^{BCP} = P \neq \pi^{BIP} = \pi^1$. As per Proposition 4.1, when $\alpha = 1$, no WOC occurs: prices are as accurate as π^1. As per Proposition 4.3, for α low enough, no agent dominates and market probability are more accurate than π^1. In this specific example, $\alpha = 0.2$ is already beyond the threshold $\bar{\alpha}$. As per Theorem 4.1, for $\alpha = .001 \approx 0$ the market probabilities distribution becomes concentrated in a small interval around π^{BCP}, which makes prices almost as accurate as the truth. If agents strongly believe that the market is accurate, then the market is indeed accurate.

5 Agents survival and WOC

The two main assumptions that enable markets to successfully combine agent beliefs are: dogmatic probabilities are diverse, and the mixing coefficient α is positive and small. In this section, we explore the role of the two assumptions on agents survival and on WOC. First, we characterize the role of α on agents survival and accuracy. Caeteris paribus, a lower α determines higher accuracy and increases an agent survival chances. Second, we appraise the role of diversity among dogmatic probabilities. Agents survival is determined not only by the accuracy of their dogmatic probabilities but also by the contribution of their dogmatic probabilities to the diversity of the market. Selection forces eliminate the agents whose dogmatic probabilities cannot be used to improve market accuracy because they are collinear with those of other agents. Lastly, we show that if agents were to use Bayes' rule to learn the weight α, then the WOC would be destroyed. Markets populated by Bayesian learners are less accurate than markets populated by agents who trust the market but are unwilling to completely relinquish their own opinion.

5.1 The role of α on the accuracy of agents

In this section, we show that giving positive weight to market probabilities weakly increases agents' accuracy. For every P, for every investment strategy adopted by other agents, and for every dogmatic probability π^i, agent i's beliefs are (weakly) more accurate if he gives positive weight to market probabilities than if he does not.

Proposition 5.1. *Under **A1-A3**, for all agents $i \in \mathcal{I}$ and for all $\alpha^i \in (0,1)$*

$$\bar{d}(P||p^i) \leq \bar{d}(P||\pi^i) \ \ P\text{-}a.s.;$$

with equality if $\lim_{t\to\infty} p^M(\cdot|\sigma^{t-1}) = \pi^i(\cdot)$.

Proof. See Appendix A. □

A skeptical reader might argue that if an agent has the correct beliefs, then he should not form beliefs according to (2.2). Proposition 5.1 tells us that, even in this case, the average accuracy of an agent is not reduced. The reason is that if agent i's dogmatic probabilities are correct, he dominates and price probability converges to the truth. Because convergence of price probability is fast, his average accuracy is not affected. In terms of consumption paths, an agent with correct dogmatic probabilities who adopts our rule has a slower growth rate of consumption-shares than that of an agent with correct beliefs who does not rely on market probabilities. However, the consumption-share volatility of the former is lower than that of the latter.

Remark: The proof of Proposition 5.1 does not require π^i or P to be i.i.d. measures. Thus, π^i could be chosen to be a learning process representing all the information that agent i can process rationally. Proposition 5.1 implies that, even in this case, each agent's accuracy is weakly increased by mixing it with market probabilities. The reason is that rule 2.2 represents a hedge against model misspecification. If π^i represents rational learning according to the correct model and using all available information, agent i's average accuracy is not diminished by mixing with market probabilities since market probability converges to π^i fast because he dominates. Otherwise, agent i's belief accuracy increases by mixing with market probabilities because his subjective probabilistic model of the world is incorrect and no amount of information suffices to make his beliefs rational.

5.2 The role of α on agents survival

Although there is a link between agent accuracy and survival, Proposition 5.1 cannot be used to make direct claims on agents survival. The reason is that changing even a single α^i alters all the properties of the equilibrium, including

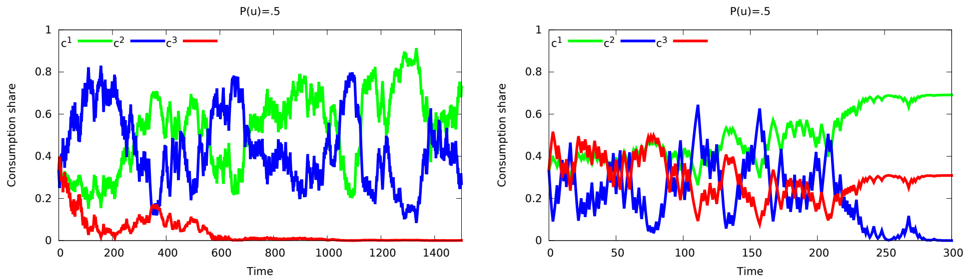

Figure 3: Consumption-share dynamics in a market with two states, three agents with $[\pi^1(u), \pi^2(u), \pi^3(u)] = [.4, .7, .4]$ and truth $P(u) = .5$. [Left]: $[\alpha^1, \alpha^2, \alpha^3] = [.2, .2, .5]$. When agent 2 is mixing, WOC occurs. Market probabilities are more accurate than $\pi^1 = \pi^3 = \pi^{BIP}$ and agent 3 vanishes because he mixes less than agent 1. [Right]: $[\alpha^1, \alpha^2, \alpha^3] = [.2, 1, .5]$. When agent 2 is not mixing, no WOC occurs. Market probabilities are as accurate as $\pi^1 = \pi^3 = \pi^{BIP}$ and agent 3 survives.

the beliefs of all agents. Here we provide a more direct argument showing that mixing with market probabilities gives a (weakly) monotonic survival advantage to an agent. If two agents have identical dogmatic probabilities, then the agent who believes less in market accuracy vanishes whenever WOC occurs.

Proposition 5.2. *Under **A1-A3**, when WOC occurs, agent j vanishes if there exists an agent i such that $\pi^j = \pi^i$ and $\alpha^i < \alpha^j$.*

Proof. See Appendix A. □

If WOC occurs, price probabilities are more accurate than all dogmatic probabilities. Therefore, giving more weight to market probability monotonically increases the accuracy of agent beliefs, thus providing an evolutionary advantage [Figure 3, left]. Conversely, there might be no accuracy gain when there is no WOC. For example, consider two agents with identical dogmatic probabilities which coincide with π^{BIP}. If no WOC occurs, market probabilities converge to π^{BIP} fast and both agents survive because their beliefs become identical [Figure 3, right].

5.3 The role of belief diversity on agents survival

How important is it to have accurate dogmatic probabilities? The diversity requirement, $\pi^{BIP} \neq \pi^{BCP}$, indicates that the answer to this question is not that simple. Agents survival is determined not only by the accuracy of their dogmatic probabilities but also by the contribution of their dogmatic probabilities to the diversity of the market. The next proposition shows that the market selects against inaccurate traders whose beliefs do not contribute to market accuracy.

Proposition 5.3. *Under **A1-A3**, agent j vanishes if there exist agent i and $\gamma \in (0,1)$ such that*

$$(i)\ \pi^i = (1-\gamma)P + \gamma\pi^j \quad \textit{and} \quad (ii)\ \alpha^i \leq \alpha^j.$$

Proof. See Appendix A. □

Condition (i) tells us that selection forces eliminate agents with redundant opinions from the market. Agents whose dogmatic probabilities are a convex combination of the truth and another agent's beliefs vanish because their beliefs do not contribute to market diversity. Condition (ii) requires that the agent with the most accurate dogmatic probabilities mixes at least as much as the least accurate agent. This condition prevents the case in which market probabilities get closer to the true measure than π^i and agent j becomes more accurate than agent i by giving more weight to market probabilities.

Proposition 5.3 implies a relation between the maximal number of surviving agents and the dimensionality of the state space: the maximal number of surviving agents is at least as large as the number of states. In markets with two states, the maximal number of surviving agents is two because agents can add diversity to market probabilities only along one dimension. The surviving agent candidates are the ones with the most accurate beliefs on the two sides of the true probability —

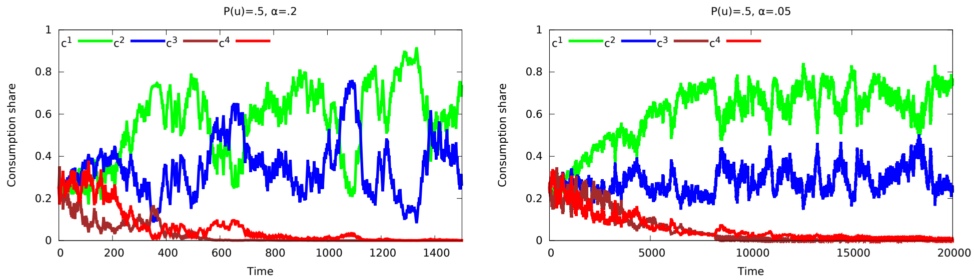

Figure 4: Consumption-share dynamics in a two-state economy with $P(u) = .5$ and four agents with dogmatic probabilities $[\pi^1(u), \pi^2(u), \pi^3(u), \pi^4(u)] = [.4, .7, .2, .8]$ and identical mixing coefficient $\alpha = .2$ [left] and $\alpha = .05$ [right]. Irrespective of the value of the mixing coefficients, only the least pessimistic and the least optimistic agents survive.

that is, the least pessimistic among the pessimistic agents and the least optimistic among the optimistic agents.

Corollary 5.1. *Under **A1-A3**, if $\mathcal{S} = \{u, d\}$, all agents have the same α, and WOC occurs, then only two agents survive. These are $\underline{i} = argmax_{i\in\mathcal{I}}\{\pi^i(u) < P(u)\}$ and $\bar{i} = argmin_{i\in\mathcal{I}}\{\pi^i(u) > P(u)\}$.*

Proof. Application of Proposition 5.3. □

Figure 4 illustrates Corollary 5.1. Two cases are presented, one in which all agents use $\alpha = .2$ [left panel] and one in which all agents use $\alpha = 0.05$ [right panel]. The simulation shows that in both cases the agents with more extreme dogmatic probabilities vanish. Although a higher weight on market probabilities makes the agents with extreme dogmatic probabilities more accurate, and thus slows down their vanishing rate considerably, in both cases, only the agents with more accurate dogmatic probabilities survive.

We conclude with a corollary showing that an agent who does not rely on market probabilities vanishes whenever WOC occurs. The result illustrates the trade-off between the value of α^i and the accuracy of agent i's dogmatic probabilities. It shows that relying on market probabilities can compensate for having less accurate dogmatic probabilities than other agents (see Figure 5 for an illustration).

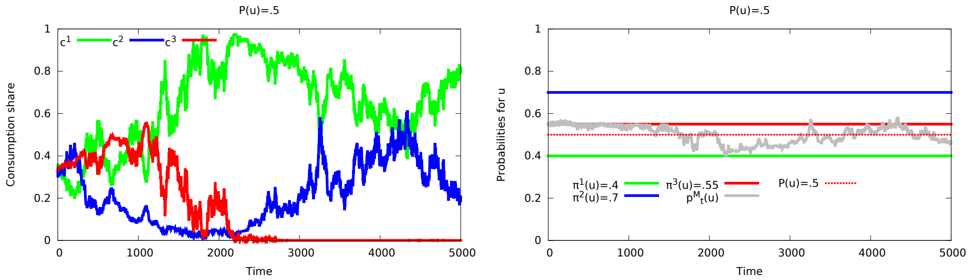

Figure 5: Consumption-share [left] and market probability dynamics [right] in a two-state, three-agent economy with $P(u)$=.5, $[\pi^1(u), \pi^2(u), \pi^3(u)]$=[.4, .7, .55] and $[\alpha^1, \alpha^2, \alpha^3] = [.1, .1, 1]$. WOC occurs because the beliefs of agents 1 and 2 are diverse and they give enough weight to market probabilities. Although agent 3 has the most accurate dogmatic probability, he vanishes because he does not believe in market accuracy.

Corollary 5.2. *Under **A1-A3**, if $\alpha^i = 1$ and WOC occurs, then agent i vanishes.*

Proof. These inequalities hold P-a.s.:
$\bar{d}(P||p^M) <^{\text{WOC occurs}} \bar{d}(P||\pi^{BIP}) \leq \bar{d}(P||\pi^i) =^{\alpha^i=1 \Leftrightarrow \pi^i = p^i} \bar{d}(P||p^i) \Rightarrow^{By\ Prop.3.2}\ i$ vanishes. $\square$

5.4 No WOC with Bayesian traders

In the previous section, we showed that a small α favors agents survival and it makes the market more accurate. Here we add a layer of sophistication to our belief formation rule. We discuss the case in which agents use Bayes' rule to learn how much weight to give to the market probability component of their beliefs.

Definition 5.1. *For all $i \in \mathcal{I}$, agent i beliefs are* Bayesian *if*

$$\forall (t, \sigma), \quad p^i(\sigma_t|\sigma^{t-1}) = \left(1 - \alpha^i_{t-1}(\sigma^{t-1})\right) p^M(\sigma_t|\sigma^{t-1}) + \alpha^i_{t-1}(\sigma^{t-1})\pi^i(\sigma_t)$$

where $\alpha^i_0 \in (0,1)$ and $\alpha^i_{t-1}(\sigma^{t-1}) := \frac{\alpha_0 \pi^i(\sigma^{t-1})}{(1-\alpha_0)p^M(\sigma^{t-1}) + \alpha_0 \pi^i(\sigma^{t-1})}$.

This layer of sophistication favors agents survival but destroys the WOC. In economies in which all agents use Bayes' rule to update the relative weight they attach to market probabilities, all agents survive, but no WOC occurs. Instead, the beliefs of all agents coordinate on the most accurate dogmatic probabilities.

Proposition 5.4. *Under **A1-A3**, if all agents' beliefs are Bayesian*

(a) all agents in the market survive in every sequence;

(b) no WOC occurs and $\lim_{t\to\infty} p^M(\cdot|\sigma^{t-1}) = \pi^{BIP}(\cdot)$, *$P$-a.s..*

Proof. See Appendix A. □

In the first step of the proof, we show that market probabilities are qualitatively equivalent to the probabilities obtained via Bayes' rule from a prior on the set of dogmatic probabilities. Because the support of market probability coincides with the set of dogmatic probabilities, the market is as accurate as the most accurate dogmatic probability (Berk, 1966; Marinacci and Massari, 2017). The agent with the most accurate dogmatic probabilities learns that his model is more accurate than the market and eventually uses only his dogmatic probabilities to form beliefs.[8] All other agents learn that the market is more accurate than their dogmatic probabilities and eventually their beliefs coincide with those of the market. Because the convergence is fast, no agent vanishes. Because all agent beliefs converge to the most accurate dogmatic probability, there is no WOC.

6 Conclusion

MSH and WOC can be reconciled in a dynamic economy where agent beliefs are diverse and partially learn from prices. When agents strongly believe in market accuracy, and the market has diverse opinions, a virtuous self-fulling prophecy occurs. Although no agent knows the truth, market selection forces endogenously

[8]More precisely, his model is more accurate than the market's in every finite horizon but coincides with that of the market asymptotically.

generate consumption-share dynamics which determine market probabilities that are almost as accurate as the most accurate combination of agents' opinions that is achievable. This positive result is destroyed if agents learn in a Bayesian fashion whether their dogmatic probability is more accurate than that of the market. In economies in which all agents use Bayes' rule to update the relative weight they attach to market probabilities, all agents survive, but no WOC occurs.

A Appendix

Lemma A.1. *Under **A1-A2**, if agents' beliefs are as in definition* (2.2) *with arbitrary* $\alpha^i \in (0,1)$ *then **A3** is satisfied and* $\forall (t,\sigma), \forall j \in \mathcal{I} \cup M, p^j(\sigma_t|\sigma^{t-1}) \in Conv(\pi^1, ..., \pi^I)$*; where* $Conv(\pi^1, ..., \pi^I)$ *is the Convex Hull of the set* $\{\pi^1, ..., \pi^I\}$.

Proof. Substituting $p^i(\sigma_t|\sigma^{t-1})$ in the equilibrium price equation (Eq.2.1),

$$\forall (t,\sigma),\ p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} \left[(1-\alpha^i)p^M(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t)\right] c^i_{t-1}(\sigma).$$

Rearranging,

$$\forall (t,\sigma),\ p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} \pi^i(\sigma_t) \frac{\alpha^i c^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}} \alpha^i c^i_{t-1}(\sigma)} \in Conv(\pi^1, ..., \pi^I). \tag{A.1}$$

$p^i(\sigma_t|\sigma^{t-1}) \in Conv(\pi^1, ..., \pi^I)$ because it is the convex combination of two points in $Conv(\pi^1, ..., \pi^I)$. **A3** is satisfied because $p^i(\sigma_t|\sigma^{t-1}) \in Conv(\pi^1, ..., \pi^I)$ and π^i being strictly positive $\forall\, i \in \mathcal{I} \Rightarrow p^i(\sigma_t|\sigma^{t-1}) \neq 0\ \forall (t,\sigma)$ and $\forall\, i \in \mathcal{I}$. ☐

Proof of Proposition 3.2

Proof.

$$
\begin{aligned}
(a):\forall i\in\mathcal{I}, p^M(\sigma^t) &=^{By\ Eq.2.1}\sum_{i\in\mathcal{I}}p^i(\sigma^t)c_0^i\\
\Rightarrow\quad & \ln p^M(\sigma^t)\geq \ln p^i(\sigma^t)+\ln c_0^i\\
\Rightarrow\quad & \frac{1}{t}\ln\frac{P(\sigma^t)}{p^M(\sigma^t)}\leq\frac{1}{t}\ln\frac{P(\sigma^t)}{p^i(\sigma^t)}-\frac{1}{t}\ln c_0^i\\
\Rightarrow\quad & \lim_{t\to\infty}\left[\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^M(\sigma_\tau|\sigma^{\tau-1})}-\sum_{\tau=1}^{t}d(P||p_\tau^M)\right]+\frac{1}{t}\sum_{\tau=1}^{t}d(P||p_\tau^M)\right]\\
&\leq\lim_{t\to\infty}\left[\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^i(\sigma_\tau|\sigma^{\tau-1})}-\sum_{\tau=1}^{t}d(P||p_\tau^i)\right]+\frac{1}{t}\sum_{\tau=1}^{t}d(P||p_\tau^i)-\frac{1}{t}\ln c_0^i\right]\\
\Rightarrow^*\quad & \bar{d}(P||p^M)\leq\bar{d}(P||p^i)\quad P\text{-a.s..}
\end{aligned}
$$

$(*)$ follows from noticing that, for $j=i,M$, $\lim_{t\to\infty}\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^j(\sigma_\tau|\sigma^{\tau-1})}-\sum_{\tau=1}^{t}d(P||p_\tau^j)\right]=0$, P-a.s. as implied by the Strong Law of Large Number for Martingale Differences (SLLNMD)(as shown also in Sandroni, 2000).

(b): We proceed by proving the contrapositive statement: $\bar{d}(P||p^i)>\bar{d}(P||p^M)$ P-a.s. $\Rightarrow$ agent i vanishes — the opposite inequality is ruled out by (a).
Note that, for all (t,σ), the FOC and market clearing imply,

$$
\begin{aligned}
c_t^i(\sigma) &= \frac{p^i(\sigma^t)}{p^M(\sigma)}c_0^i\\
\Leftrightarrow\frac{1}{t}\ln c_t^i(\sigma) &= \frac{1}{t}\ln\frac{p^i(\sigma^t)}{p^M(\sigma^t)}+\frac{1}{t}\ln c_0^i\\
&=\frac{1}{t}\left[\ln\frac{P(\sigma^t)}{p^M(\sigma^t)}-\ln\frac{P(\sigma^t)}{p^i(\sigma^t)}\right]+\frac{1}{t}\ln c_0^i
\end{aligned}
$$

Proceeding as in (a), we obtain $\lim_{t\to\infty}\frac{1}{t}\ln c_t^i(\sigma)=\bar{d}(P||p^M)-\bar{d}(P||p^i)$ P-a.s., by the SLLNMD

$$
\begin{aligned}
\text{Therefore, } \bar{d}(P||p^i)>\bar{d}(P||p^M)\ P\text{-a.s.} &\Rightarrow\frac{1}{t}\ln c_t^i(\sigma)<0, P\text{-a.s.}\\
&\Leftrightarrow \ln c_t^i(\sigma)\to-\infty, P\text{-a.s.}\\
&\Leftrightarrow c_t^i\to 0:\text{ agent } i \text{ vanishes.}
\end{aligned}
$$

□

Proof of Proposition 3.3

Proof. $\bar{d}(P||p^M)\leq^{By\ Prop.3.2}\bar{d}(P||p^{BIP})\leq^{By\ Prop.5.1}\bar{d}(P||\pi^{BIP})$, P-a.s..
Moreover, $\forall\sigma,\bar{d}(P||\pi^{BCP})\leq\bar{d}(P||p^M)$ because $\pi^{BCP}:=\underset{p\in Conv(\pi^1,\dots,\pi^I)}{\text{argmin}}\ d(P||p)$ and $\forall(t,\sigma),p_t^M\in^{By\ Lem.A.1}Conv(\pi^1,...,\pi^I)$. □

Proof of Proposition 4.1

Proof. We prove the contrapositive statement:

$$\alpha^i = 1 \ \forall i \in \mathcal{I} \Rightarrow \bar{d}(P||p^M) = \bar{d}(P||\pi^{BIP}) \ P\text{-a.s.} \Leftrightarrow \text{no WOC}.$$

First note that $\alpha^i = 1 \ \forall i \in \mathcal{I} \Rightarrow p^i(\sigma^t) = \pi^i(\sigma^t), \forall i, \forall(\sigma, t)$. The result follows by noticing that by Proposition 3.1 only the most accurate agent, π^{BIP}, survives and by Proposition 3.2 the market is as accurate as every agent that survive. $\square$

Proof of Proposition 4.2

Proof. We prove the contrapositive statement:
$\pi^{BCP} = \pi^{BIP} \Rightarrow \bar{d}(P||p^M) = \bar{d}(P||\pi^{BIP})$ P-a.s. $\Leftrightarrow$ no WOC,
which follows noticing that, for every choice of $\alpha^i \in (0, 1]$ for all $i \in \mathcal{I}$,
$\bar{d}(P||\pi^{BIP}) \geq^{By\ Prop.5.1} \bar{d}(P||p^{BIP}) \geq^{By\ Prop.3.2} \bar{d}(P||p^M) \geq \bar{d}(P||\pi^{BCP}) = \bar{d}(P||\pi^{BIP})$.
The last inequality follows because $\forall(t, \sigma), p^M_t \in^{By\ Lem.A.1} Conv(\pi^1, ..., \pi^I) \Rightarrow \forall(t, \sigma), d(P||p^M_t) \geq d(P||\pi^{BCP})$. Summing and averaging over t gives the result.

$\square$

In order to prove Proposition 4.3 we need these two Lemmas.

Lemma A.2. *Under* ***A1-A3****,*

$$\pi^{BIP} \neq \pi^{BCP} \Rightarrow \exists\, j \in \mathcal{I} : \nabla d(P||\pi^{BIP}) \cdot (\pi^j - \pi^{BIP}) < 0$$

Proof. By contradiction suppose that for all $i \in \mathcal{I}$

$$\nabla d(P||\pi^{BIP}) \cdot (\pi^i - \pi^{BIP}) \geq 0 \tag{A.2}$$

By linearity of the scalar product, eq. (A.2) implies that for every $\bar{\pi} \in Conv(\pi^1, ..., \pi^I)$

$$\nabla d(P||\pi^{BIP}) \cdot (\bar{\pi} - \pi^{BIP}) \geq 0.$$

As a result, $\pi^{BCP} \in Conv(\pi^1, ..., \pi^I)$ implies

$$\nabla d(P||\pi^{BIP}) \cdot (\pi^{BCP} - \pi^{BIP}) \geq 0. \tag{A.3}$$

Convexity and smoothness of the K-L divergence imply that for all γ

$$d(P||\pi^{BIP} + \gamma(\pi^{BCP} - \pi^{BIP})) \geq d(P||\pi^{BIP}) + \gamma \nabla d(P||\pi^{BIP}) \cdot (\pi^{BCP} - \pi^{BIP}).$$

In particular, taking $\gamma = 1$ and using eq. (A.3) gives

$$d(P||\pi^{BCP}) \geq d(P||\pi^{BIP}),$$

a contradiction. $\square$

Lemma A.3. *Under **A1-A3**,*

$\pi^{BIP} \neq \pi^{BCP} \Rightarrow \exists\, \bar{\alpha} \in (0,1)$ *such that* $\forall i \in \mathcal{I}, \alpha^i \in (0, \bar{\alpha})$ *implies* $\lim_{t\to\infty} p_t^M \neq \pi^{BIP}$ *P-a.s..*

Proof. By contradiction, assume

(i) $\pi^{BIP} \neq \pi^{BCP}$

and

(ii) $\forall \bar{\alpha} \in (0,1)$ and $\forall i \in \mathcal{I}$ $\alpha^i \in (0, \bar{\alpha})$ implies $\forall \epsilon > 0\ \exists T$ such that $\forall t > T \quad ||p_t^M - \pi^{BIP}|| < \epsilon$ P-a.s..

We shall prove two intermediate results.

First, $(ii) \Rightarrow \bar{d}(P||\pi^{BIP}) = \bar{d}(P||p^M) \quad P$-a.s. (A.4)

By continuity of the K-L divergence,

$$\lim_{t\to\infty} p_t^M = \pi^{BIP} \ \ P\text{-a.s.} \Rightarrow \lim_{t\to\infty} d(P||p_t^M) - d(P||\pi^{BIP}) = 0 \ \ P\text{-a.s..}$$

Let $g_t = \frac{1}{t}\sum_{\tau=1}^{t}\left[d(P||p_\tau^M) - d(P||\pi^{BIP})\right]$.
Cesàro theorem (e.g. Williams, 1991, pg.116) implies that $\bar{d}(P||\pi^{BIP}||P) - \bar{d}(P||p^M||P) := \lim_{t\to\infty} g_t = 0$ P-a.s..

Second, (i) and $(ii) \Rightarrow \exists j \in \mathcal{I} : \bar{d}(P||\pi^{BIP}) > \bar{d}(P||p^j) \quad P$-a.s. (A.5)

By Lemma A.2, (i) guarantees that there exists a $j \in \mathcal{I}$ such that

$$\nabla d(P||\pi^{BIP}) \cdot (\pi^j - \pi^{BIP}) < 0. \tag{A.6}$$

Rewrite p_t^j as $p_t^j = \pi^{BIP} + \alpha^j(\pi^j - \pi^{BIP}) + (1-\alpha^j)(p_t^M - \pi^{BIP})$.
Computing the K-L divergence of beliefs p_t^j and expanding around π^{BIP} gives

$$\begin{aligned} d(P||p_t^j) &= d(P||\pi^{BIP} + \alpha^j(\pi^j - \pi^{BIP}) + (1-\alpha^j)(p_t^M - \pi^{BIP}))) \\ &= d(P||\pi^{BIP} + \alpha^j((\pi^j - \pi^{BIP}) + \frac{(1-\alpha^j)}{\alpha^j}(p_t^M - \pi^{BIP}))) \\ &= d(P||\pi^{BIP}) + \alpha^j \nabla d(P||\pi^{BIP}) \cdot \left((\pi^j - \pi^{BIP}) + \frac{(1-\alpha^j)}{\alpha^j}(p_t^M - \pi^{BIP})\right) + o(\alpha^j). \end{aligned}$$

The above equality, together with (A.6) and (ii) implies that P-a.s. there exists a $\bar{\alpha}$ such that for all $\alpha^j < \bar{\alpha}$ there exists a $T(\alpha^j)$ and a $\delta > 0$ such that for all $t > T(\alpha^j)$

$$d(P||p_t^j) < d(P||\pi^{BIP}) - \delta.$$

(A.5) is now proven by summing over t and averaging.

Finally, the Lemma follows from noticing that (A.4) and (A.5) imply that

$$\exists j \in \mathcal{I}, \exists \bar{\alpha} > 0 : \alpha \in (0, \bar{\alpha}) \Rightarrow \bar{d}(P||p^M) = \bar{d}(P||\pi^{BIP}) > \bar{d}(P||p^j) \;\; P\text{-a.s.},$$

Contradicting Prop. 3.2 (a) which states that $\forall j \in \mathcal{I}, \bar{d}(P||p^M) \leq \bar{d}(P||p^j)$ P-a.s.. □

Proof of Proposition 4.3

Under **A1-A3**,

$$\pi^{BIP} \neq \pi^{BCP} \Rightarrow \exists \bar{\alpha} \in (0,1) : \forall i \in \mathcal{I}, \alpha^i < \bar{\alpha} \Rightarrow \bar{d}(P||p^M) < \bar{d}(P||\pi^{BIP}) \quad P\text{-a.s.}.$$

Proof. Under the stated assumptions

$$\bar{d}(P||p^M) \leq^{By\ Prop. 3.2} \bar{d}(P||p^{BIP}) \leq^{By\ Prop. 5.1} \bar{d}(P||\pi^{BIP}) \quad P\text{-a.s.}.$$

By strict convexity of the K-L divergence, these inequalities are strict unless:
(*i*) $\lim_{t\to\infty} p_t^M = \pi^{BIP}$; or (*ii*) $\forall i \in \mathcal{I}\alpha = 0$; or (*iii*) $\forall i \in \mathcal{I}\alpha^i = 1$ — which implies $\bar{d}(P||p^M) = \bar{d}(P||\pi^{BIP})$.
The result follows by noticing that cases (*ii*) and (*iii*) are ruled out by assumption; and that case (*i*) is ruled out by Lemma A.3 when α is small enough. □

Proof of Proposition 5.1

Proof. $\forall(t, \sigma)$,

$$\begin{aligned} d(P||p_t^i) &= d(P||(1-\alpha^i)p_t^M + \alpha^i\pi^i)) \\ &\leq^a (1-\alpha^i)d(P||p_t^M) + \alpha d(P||\pi^i) && \text{; by strict convexity of } d(P||\cdot) \\ \Rightarrow \quad & \bar{d}(P||p^i) \leq (1-\alpha^i)\bar{d}(P||p^M) + \alpha^i\bar{d}(P||\pi^i) && \text{; summing and averaging over } t \\ \Rightarrow \quad & \bar{d}(P||p^i) \leq \bar{d}(P||\pi^i) \; P\text{-a.s.} && \text{; Because } \bar{d}(P||p^M) \leq^{by\ Prop. 3.2} \bar{d}(P||p^i) \end{aligned}$$

We now prove that equality holds if $p_t^M \to \pi^i$.
Given that inequality (a) is strict unless $p_t^M(\sigma) = \pi^i$, continuity of $d(P||\cdot)$ and $p_t^M(\sigma) \to \pi^i \Rightarrow d(P||p_t^M) \to d(P||\pi^i)$. Applying the Cesàro theorem (e.g. Williams, 1991, pg.116) to the definition of $\bar{d}(P||p^i)$ leads to $\bar{d}(P||p^i) = \bar{d}(P||\pi^i)$. □

Proof of Proposition 5.2

Proof. By assumption WOC occurs, $\pi^i = \pi^j = \pi$ and $\alpha^i < \alpha^j$. Thus, $\forall(t, \sigma)$,

$$\begin{aligned} & p_t^i = (1-\lambda)p^M + \lambda p^j && \text{; with } \lambda = \frac{\alpha^i}{\alpha^j} \\ \Rightarrow & d(P||p_t^i) \leq^a (1-\lambda)d(P||p_t^M) + \lambda d(P||p_t^j) && \text{; By strict convexity of the K-L divergence} \\ \Rightarrow & \bar{d}(P||p^i) \leq^b (1-\lambda)\bar{d}(P||p^M) + \lambda\bar{d}(P||p^j) && \text{; Rearranging, summing and averaging over } t \end{aligned}$$

If inequality (b) is strict, rearranging we obtain $\bar{d}(P||p^i) < \bar{d}(P||p^j)$ and agent j vanishes by Proposition 3.1.
Inequality (b) is strict by the following argument.
(a) holds with equality iff $p_t^i = p_t^j$ at t; that is, iff $p_t^i = p_t^j = p_t^M = \pi$ at t. Thus, by Cesàro theorem (b) holds with equality only if p_t^i, p_t^j, p_t^M converge to π. Which contradicts our initial assumption that WOC occurs. □

Proof of Proposition 5.3

Proof. By assumption, $\alpha^i \leq \alpha^j$ and $\pi^i = (1-\gamma)P + \gamma\pi^j$ for a $\gamma \in (0,1)$.
For all t, let $\bar{p}_t := (1-\alpha^i)p_t^M + \alpha^i P$. Using the decomposition of π^i we rewrite p_t^i as

$$\begin{aligned} p_t^i &= (1-\alpha^i)(1-\gamma+\gamma)p_t^M + \alpha^i(1-\gamma)P + \alpha^i\gamma\pi^j \\ &= (1-\gamma)\bar{p}_t + \gamma\left((1-\alpha^i)p_t^M + \alpha^i\pi^j\right) \\ &= (1-\gamma)\bar{p}_t + \gamma\left((1-\lambda)p_t^M + \lambda p_t^j\right) \qquad ; \text{ with } \lambda = \frac{\alpha^i}{\alpha^j} \in (0,1). \end{aligned}$$

By construction, $\forall(t,\sigma)$, $p_t^i \neq \bar{p}_t$, thus, strict convexity of the K-L divergence ensures that $\exists\epsilon > 0$: for all (t,σ),

$$d(P||p_t^i) < (1-\gamma)d(P||\bar{p}) + \gamma(1-\lambda)d(P||p_t^M) + \gamma\lambda d(P||p_t^j) - \epsilon.$$

Summing over t and averaging leads to

$$\bar{d}(P||p^i) - \bar{d}(P||p^j) < (1-\gamma)\bar{d}(P||\bar{p}) + \gamma(1-\lambda)\bar{d}(P||p^M) + (\gamma\lambda - 1 + \gamma - \gamma)\bar{d}(P||p^j).$$

which rearranging gives

$$\begin{aligned} \bar{d}(P||p^i) - \bar{d}(P||p^j) &< (1-\gamma)\left[\bar{d}(P||\bar{p}) - \bar{d}(P||p^j)\right] + \gamma(1-\lambda)[\bar{d}(P||p^M) - \bar{d}(P||p^j)] \\ &<^a (1-\gamma)\left[\bar{d}(P||\bar{p}) - \bar{d}(P||p^j)\right] + 0 \\ &<^b (1-\gamma)\left[\bar{d}(P||p^M) - \bar{d}(P||p^j)\right] \\ &<^a 0 \quad P\text{-a.s.} \end{aligned}$$

(a) : Because $\bar{d}(P||p^M) - \bar{d}(P||p^j) \leq 0$ P-a.s., by Prop 3.2
(b) : By definition of $\bar{p}$ and convexity of the K-L, $\bar{d}(P||\bar{p}) \leq (1-\alpha)\bar{d}(P||p^M) \leq \bar{d}(P||p^M)$

Because $\bar{d}(P||p^i) - \bar{d}(P||p^j) < 0$ P-a.s., agent j vanishes by Proposition 3.1 □

Proof of Proposition 5.4

Proof.

(a) all agents survive in every sequence.

Definition 5.1 implies $\forall(\sigma,t)$:

$$\forall i\in\mathcal{I},\quad p^i(\sigma^t):=\prod_{\tau=1}^{t}p^i(\sigma_\tau|\sigma^{\tau-1})=\left(1-\alpha_0^i\right)p^M(\sigma^t)+\alpha_0^i\pi^i(\sigma^t).$$

Note that $\forall(t,\sigma),\forall i\in\mathcal{I}$, $\lim_{t\to\infty}\frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}=\lim_{t\to\infty}\frac{\beta^t\left(\left(1-\alpha_0^i\right)p^M(\sigma^t)+\alpha_0^i\pi^i(\sigma^t)\right)}{\beta^t p^M(\sigma^t)}>0$. Thus, all agents survive in all sequences because no trader satisfies Massari (2017) necessary and sufficient condition for a trader to vanish.

(b) no WOC occur and $\lim_{t\to\infty}p^M(\cdot|\sigma^{t-1})=\pi^{BIP}(\cdot)$, P-a.s..
Equation (2.1) implies $\forall(\sigma,t)$:

$$p^M(\sigma^t)=\sum_{i\in\mathcal{I}}c_0^i p^i(\sigma^t)=\sum_{i\in\mathcal{I}}\frac{c_0^i\alpha_0^i}{\sum_{i\in\mathcal{I}}c_0^i\alpha_0^i}\pi^i(\sigma^t)$$

Thus, $p^M(\sigma^t)$ is formally equivalent to the probability obtained by Bayes' rule with prior weights $\left[\frac{c_0^1\alpha_0^1}{\sum_{i\in\mathcal{I}}c_0^i\alpha_0^i},...,\frac{c_0^I\alpha_0^I}{\sum_{i\in\mathcal{I}}c_0^i\alpha_0^i}\right]$ on models $\left[\pi^1,...,\pi^I\right]$. The result follows from Berk (1966) which shows that the bayesian posterior converges a.s. to the model in the support with the lowest K-L divergence, π^{BIP}.

□

B Proof of Theorem 4.1

Let start by quoting Theorem 3.1 of Hajek (1982) and an immediate Corollary (B.1). We are going to use Corollary B.1 to show that α can be chosen small enough to put a tight probabilistic bound on the fraction of periods market probabilities are outside of an arbitrarily small neighborhood of P when $P=\pi^{BCP}$.

Let $(Y_t)_{t=0}^{\infty}$ be a sequence of real valued adopted random variables, with drift $\mathrm{E}[Y_{t+1}-Y_t|\mathcal{F}_t]$ such that for some $a\in(-\infty,+\infty)$ and $\rho\in(0,1),\eta\in(0,\infty),D<\infty$:

$$\begin{aligned}D0:&\quad \mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t]\le D\quad for\quad t\ge 0\,,\\ D1:&\quad \mathrm{E}[e^{\eta(Y_{t+1}-Y_t)};\,Y_t>a|\mathcal{F}_t]\le\rho\quad for\quad t\ge 0\,,\\ D2:&\quad \mathrm{E}[e^{\eta(Y_{t+1}-a)};\,Y_t\le a|\mathcal{F}_t]\le D\quad for\quad t\ge 0.\end{aligned}$$

Theorem B.1. *(Hajek, 1982). Assume conditions $D0$, $D1$, and $D2$. For any $\epsilon>0$ and $b>a$, there exist constants K and $\delta<1$ such that*

$$P\left\{\frac{1}{T}\sum_{t=1}^{T}I_{\{Y_t<b\}}\le\rho_0(1-\epsilon)\right\}\le K\delta^T\,,\tag{B.1}$$

where $\rho_0 = 1 - \frac{1}{1-\rho} D e^{\eta(a-b)}$.

Corollary B.1. *Assume conditions D0, D1, and D2. For any $\epsilon > 0$ and $b > a$, there exist constants K and $\delta < 1$ such that*

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{Y_t<b\}} \geq \rho_0(1-\epsilon)\right\} \geq 1 - K\delta^T.$$

Proof. $P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{Y_t<b\}} \geq \rho_0(1-\epsilon)\right\} = 1 - P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{Y_t<b\}} \leq \rho_0(1-\epsilon)\right\} \geq^{by\ B.1} 1 - K\delta^T$.
□

Reminder of our notation	
$S = \{u, d\}$	*The state space only has two states*
P	*True probability*
$\mathcal{I}$	*Set of I agents*
π^i	*Dogmatic probabilities of agent $i \in \mathcal{I}$*
$p^M(\sigma_t\|\sigma^{t-1})$	*Market probability of state σ_t in σ^{t-1}*
$p^i(\sigma_t\|\sigma^{t-1})$	*Agent i's belief of state σ_t in σ^{t-1}*
$c_t^i(\sigma)$	*Agent i's consumption in (t, σ)*
$\alpha^i = \alpha \in (0, 1]$	*Agent i's mixing coefficient*
$\begin{cases} p^M(\sigma_t\|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} c_{t-1}^i(\sigma) p^i(\sigma_t\|\sigma^{t-1}) \\ c_t^i(\sigma) = c_{t-1}^i(\sigma) \frac{p^i(\sigma_t\|\sigma^{t-1})}{p^M(\sigma_t\|\sigma^{t-1})} \\ p^i(\sigma_t\|\sigma^{t-1}) = (1-\alpha) p^M(\sigma_t\|\sigma^{t-1}) + \alpha\pi^i(\sigma_t) \end{cases}$	*Consumption-share and beliefs dynamics*

By Equation (A.1), if agents use identical weights α, then market probabilities are

$$p^M(\sigma_{t+1}|\sigma^t) = \sum_{i\in\mathcal{I}} c_t^i(\sigma)\pi^i(\sigma_{t+1}). \tag{B.2}$$

To ease notation we focus on state u and define $P := P\{\sigma_{t+1} = u\}$, $\pi^{BCP} := \pi^{BCP}\{\sigma_{t+1} = u\}$, $\pi^i := \pi^i(\sigma_{t+1} = u|\sigma^t)$, $p_{t+1}^M := p^M(\sigma_{t+1} = u|\sigma^t)$ and $c_t^i := c_t^i(\sigma)$.
Because the economy only has two states, Corollary 5.1 implies that at most two agents survive and we can focus WLOG on the case in which there are only two agents: $l = \underline{i}$ and $r = \overline{i}$ with $\pi^l < P = P^{BCP} < \pi^r$.[9] Our result depends on the evolution of the log-ratio of the consumption-shares between these two agents, that is, the real adapted process $Y_t = \log\left(\frac{c_t^r}{c_t^l}\right)$.

[9]The current proof can be modified to directly cover the case of more than two agents by replacing π^r and π^l by $\pi_t^R := \sum_{p_t^i > P} \pi^i c_{t-1}^i(\sigma)$ and $\pi_t^L := \sum_{p_t^i < P} \pi^i c_{t-1}^i(\sigma)$; $c_t^r(\sigma)$ and $c_t^l(\sigma)$ with $c_t^R(\sigma) := \sum_{p_t^i > P} c_{t-1}^i(\sigma)$ and $c_t^L(\sigma) := \sum_{p_t^i < P} c_{t-1}^i(\sigma)$ and providing a bound on the difference $\pi_t^R - \pi_t^L$ that holds uniformly for every consumption-share distribution within agents in L and R.

We introduce a number of Lemmas to show that, as $\alpha \to 0$, $(Y_t)_{t=0}^{\infty}$ satisfies conditions $D0$, $D1$, and $D2$ in such a way to ensure that p_{t+1}^M stays most of the time arbitrarily close to P.

Lemma B.1. *Under **A1-A3**, for all $t \geq 0$, for all $\eta > 0$,*

$$\mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t] = P\left(1+\alpha\frac{\pi^r-\pi^l}{\alpha\pi^l+(1-\alpha)p_{t+1}^M}\right)^{\eta} + (1-P)\left(1-\alpha\frac{\pi^r-\pi^l}{\alpha(1-\pi^l)+(1-\alpha)(1-p_{t+1}^M)}\right)^{\eta}.$$

Proof. By definition of $(Y_t)_{t=0}^{\infty}$,

$$Y_{t+1}-Y_t = \log\frac{c_{t+1}^r}{c_{t+1}^l} - \log\frac{c_t^r}{c_t^l} = \log\frac{p^r(\sigma_{t+1}|\sigma_t)}{p^l(\sigma_{t+1}|\sigma_t)}.$$

Thus,

$$\begin{aligned}
Y_{t+1}|_{\sigma_{t+1}=u} - Y_t &= \log\left(\frac{\alpha\pi^r+(1-\alpha)p_{t+1}^M}{\alpha\pi^l+(1-\alpha)p_{t+1}^M}\right);\\
Y_{t+1}|_{\sigma_{t+1}=d} - Y_t &= \log\left(\frac{\alpha(1-\pi^r)+(1-\alpha)(1-p_{t+1}^M)}{\alpha(1-\pi^l)+(1-\alpha)(1-p_{t+1}^M)}\right).
\end{aligned}$$

Computing the expected value

$$\begin{aligned}
\mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t] &= P\left(\frac{\alpha\pi^r+(1-\alpha)p_{t+1}^M}{\alpha\pi^l+(1-\alpha)p_{t+1}^M}\right)^{\eta} + (1-P)\left(\frac{\alpha(1-\pi^r)+(1-\alpha)(1-p_{t+1}^M)}{\alpha(1-\pi^l)+(1-\alpha)(1-p_{t+1}^M)}\right)^{\eta}\\
&= P\left(1+\alpha\frac{\pi^r-\pi^l}{\alpha\pi^l+(1-\alpha)p_{t+1}^M}\right)^{\eta} + (1-P)\left(1-\alpha\frac{\pi^r-\pi^l}{\alpha(1-\pi^l)+(1-\alpha)(1-p_{t+1}^M)}\right)^{\eta}.
\end{aligned}$$

□

Lemma B.2. *Under **A1-A3**, for all $t \geq 0$, let $\eta = \frac{x}{\alpha}$, with $x \in [\alpha, 1]$, then there exists a $B < \infty$, independent of η and α, such that:*

$$\mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t] \leq 1 - x(p_{t+1}^M - P)\frac{\pi^r-\pi^l}{p_{t+1}^M(1-p_{t+1}^M)} + x^2\frac{B}{2}.$$

Proof. For $\alpha = \frac{x}{\eta}$, we can express $\mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t]$ as a function of x, η and p^M:

$$\begin{aligned}
f(x,\eta,p_{t+1}^M) &:= \mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}|\mathcal{F}_t]\\
&=^{\text{By Lem.B.1}} P\left(1+\frac{x}{\eta}\frac{\pi^r-\pi^l}{\frac{x}{\eta}(\pi^l-p_{t+1}^M)+p_{t+1}^M}\right)^{\eta} + (1-P)\left(1-\frac{x}{\eta}\frac{\pi^r-\pi^l}{\frac{x}{\eta}(p_{t+1}^M-\pi^l)+1-p_{t+1}^M}\right)^{\eta}.
\end{aligned}$$

Taylor expanding f in x around 0, $\forall p^M \in [\pi^l, \pi^r]$

$$f(x, \eta, p^M) = 1 - x(p^M - P)\frac{\pi^r - \pi^l}{p^M(1-p^M)} + x^2 \frac{B(\xi, \eta, p^M)}{2}$$

for a $\xi \in (0, x)$ and

$$B(\xi, \eta, p^M) = \left.\frac{\partial^2 f(x, \eta, p^M)}{\partial x^2}\right|_{x=\xi} = PB_{up}(\xi, \eta, p^M, \pi^l, \pi^r) + (1-P)B_{down}(\xi, \eta, p^M, \pi^l, \pi^r);$$

$$\text{with} \begin{cases} B_{up}(\xi, \eta, p^M, \pi^l, \pi^r) &= \eta(\eta-1)\left(1 + \frac{\xi}{\eta}\frac{\pi^r - \pi^l}{(\pi^l - p^M)\frac{x}{\eta} + p^M}\right)^{\eta-2}\left(\frac{\frac{\pi^r-\pi^l}{\eta}}{(\pi^l - p^M)\frac{x}{\eta} + p^M} - \frac{\frac{\xi}{\eta^2}(\pi^r - \pi^l)(\pi^l - p^M)}{((\pi^l - p^M)\frac{x}{\eta} + p^M)^2}\right)^2 + \\ & + \eta\left(1 + \frac{\xi}{\eta}\frac{\pi^r - \pi^l}{(\pi^l - p^M)\frac{x}{\eta} + p^M}\right)^{\eta-1}\left(-\frac{\frac{2}{\eta^2}(\pi^r - \pi^l)(\pi^l - p^M)}{((\pi^l - p^M)\frac{x}{\eta} + p^M)^2} + \frac{2\frac{\xi}{\eta^3}(\pi^r - \pi^l)(\pi^l - p^M)^2}{((\pi^l - p^M)\frac{x}{\eta} + p^M)^3}\right)^2 \\ B_{down}(\xi, \eta, p^M, \pi^l, \pi^r) &= B_{up}(\xi, \eta, 1 - p^M, 1 - \pi^l, 1 - \pi^r) \end{cases}.$$

The proof proceeds by finding a finite upper bound for $B_{up}(\xi, \eta, p^M, \pi^l, \pi^r)$ and $B_{down}(\xi, \eta, p^M, \pi^l, \pi^r)$ that holds for all $\xi \in [0, 1]$, $\eta \in [1, +\infty)$, and $p^M \in [\pi^l, \pi^r]$. For this purpose note that the limit for $\eta \to \infty$ is well defined,

$$\lim_{\eta \to \infty} B_{up}(\xi, \eta, p^M, \pi^l, \pi^r) = \frac{(\pi^r - \pi^l)^2}{(p^M)^2} e^{\xi\frac{(\pi^r - \pi^l)}{p^M}}.$$

Naming $y = 1/\eta$ the function $B_{up}(\xi, y = \frac{1}{\eta}, p^M, \pi^l, \pi^r)$ is continuous on the compact $[0, 1] \times [0, 1] \times [\pi^l, \pi^r]$. By the Weierstrass Theorem such a function has a maximum $\bar{B}_{up} < \infty$. In the same way B_{down} has a maximum $\bar{B}_{down}$. Defining

$$B(\pi^l, \pi^r) = \left|P\bar{B}_{up} + (1-P)\bar{B}_{down}\right| < \infty$$

we have proved that for all t and for all α and η such that $\alpha\eta = x \in [0, 1]$

$$f(x, \eta, p^M_{t+1}) \leq 1 - x(p^M_{t+1} - P)\frac{\pi^r - \pi^l}{p^M_{t+1}(1 - p^M_{t+1})} + x^2\frac{B}{2}.$$

□

Next we present a series of lemmas showing that conditions $D0$, $D1$, $D2$, holds uniformly for arbitrarily large η, as long as $\eta\alpha = x$ is kept small enough. Importantly for the proof of Theorem B.2, these lemmas requires $\eta\alpha$ to be small but allow to send $\alpha \to 0$ and thus to make η arbitrarily large.

Lemma B.3. ***D0****: Under* ***A1-A3****, there exists a $D < \infty$ such that if $\eta = \frac{x}{\alpha}$, then*

$$\forall x \in [\alpha, 1],\ \forall t \geq 0\ \ \mathrm{E}[e^{\eta(Y_{t+1} - Y_t)} | \mathcal{F}_t] < D.$$

Proof. The proof follows easily by Lemma B.2 upon choosing

$$D = \max_{x \in [\alpha,1], p^M \in [\pi^l, \pi^r]} \left\{ 1 - x(p^M - P) \frac{\pi^r - \pi^l}{p^M(1-p^M)} + x^2 \frac{B}{2} \right\}.$$

□

Lemma B.4. ***D1**: Under **A1-A3**, $\forall \epsilon_a > 0$, let $a = \log\left(\frac{P+\epsilon_a-\pi^l}{\pi^r - P - \epsilon_a}\right)$, $\bar{x}_a = \frac{4\epsilon_a(\pi^r-\pi^l)}{B}$ and $\eta = \frac{\bar{x}_a}{\alpha}$. Then, $\exists \rho(\bar{x}_a) < 1$:*

$$\forall \alpha \in (0, \bar{x}_a],\ \forall t \geq 0,\ \ \mathrm{E}\left[e^{\eta(Y_{t+1}-Y_t)};\, Y_t > a\,|\mathcal{F}_t\right] \leq \rho(\bar{x}_a) < 1.$$

Proof. By Equation (B.2), $p^M_{t+1} = \pi^l c^l_t + \pi^r c^r_t$, so that $Y_t := \log\left(\frac{c^r_t}{c^l_t}\right) = \log\left(\frac{p^M_{t+1}-\pi^l}{\pi^r - p^M_{t+1}}\right)$. Therefore, for any ϵ_a and a, on the event $\{Y_t > a\}$ the price satisfies $p^M_{t+1} > P + \epsilon_a$. Together with Lemma B.2 this implies that for all t

$$\begin{aligned}
\mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}; Y_t > a|\mathcal{F}_t] &\leq 1 - x\epsilon_a \frac{\pi^r - \pi^l}{p^M_{t+1}(1-p^M_{t+1})} + x^2\frac{B}{2} \\
&\leq 1 - 4x\epsilon_a(\pi^r - \pi^l) + x^2 \frac{B}{2} \\
&\leq 1 - \frac{4^2\epsilon_a^2(\pi^r-\pi^l)^2}{2B} \qquad ;\text{for } x = \bar{x}_a = \frac{4\epsilon_a(\pi^r-\pi^l)}{B} \\
&= \rho(\bar{x}_a) < 1.
\end{aligned}$$

□

Lemma B.5. ***D2**: Under **A1-A3**, $\forall \epsilon_a > 0$, let $a = \log\left(\frac{P+\epsilon_a-\pi^l}{\pi^r - P - \epsilon_a}\right)$ and $\eta = \frac{x}{\alpha}$. Then, there exists a $D < \infty$ such that*

$$\forall x \in [\alpha, 1],\ \forall t \geq 0\ \ \mathrm{E}[e^{\eta(Y_{t+1}-a)};\, Y_t \leq a|\mathcal{F}_t] < D.$$

Proof. Note that $g < a \Rightarrow \mathrm{E}[e^{\eta(Y_{t+1}-a)};\, Y_t = g|\mathcal{F}_t] \leq \mathrm{E}[e^{\eta(Y_{t+1}-Y_t)}; Y_t = g|\mathcal{F}_t]$.
The latter is bounded by a D for all g by Lemma B.3.

□

Lemma B.6. *In an economy with 2 states and 2 agents with diverse beliefs, for any $\epsilon_b > 0$ and $\epsilon_q \in (0,1)$ there exists $\bar{\alpha}_r$, $K < \infty$ and $\delta \in (0,1)$ such that for all $\alpha \in (0, \bar{\alpha}_r)$*

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{p^M_t < \pi^{BCP} + \epsilon_b\}} \geq 1 - \epsilon_q\right\} \geq 1 - K\delta^T$$

Proof. By Corollary B.1, under $D0$, $D1$, $D2$, for any $\epsilon \in (0,1)$ and $b > a$ there exists a constant $K < +\infty$ and $\delta < 1$ such that,

$$P\left\{\frac{1}{T}\sum_{t=0}^{T-1} I_{\{Y_t<b\}} \geq \left(1-\frac{1}{1-\rho}De^{\eta(a-b)}\right)(1-\epsilon)\right\} \geq 1-K\delta^T. \tag{B.3}$$

Let $b = \log\left(\frac{\pi^{BCP}+\epsilon_b-\pi^l}{\pi^r-\pi^{BCP}-\epsilon_b}\right)$, so that $\{Y_t < b\}$ if and only if $p^M_{t+1} < \pi^{BCP} + \epsilon_b$. Setting $a = \frac{b}{2}$, Equation (B.3) becomes

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{p_t^M<\pi^{BCP}+\epsilon_b\}} \geq \left(1-\frac{1}{1-\rho}De^{-\eta\frac{b}{2}}\right)(1-\epsilon)\right\} \geq 1-K\delta^T. \tag{B.4}$$

By Lemma B.4, taking $\alpha \in (0, \bar{x}_a]$ and $\eta = \frac{\bar{x}_a}{\alpha}$ implies

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{p_t^M<\pi^{BCP}+\epsilon_b\}} \geq \left(1-\frac{De^{-\frac{b\bar{x}_a}{2\alpha}}}{1-\rho(\bar{x}_a)}\right)(1-\epsilon)\right\} \geq 1-K\delta^T,$$

where the denominator is independent on α. Thus, for every $\epsilon_q \in (0,1)$, there exists an $\bar{\alpha}_r$ and an ϵ such that for every $\alpha \in (0, \bar{\alpha}_r)$

$$\left(1-\frac{De^{-\frac{b\bar{x}_a}{2\alpha}}}{1-\rho(\bar{x}_a)}\right)(1-\epsilon) \geq (1-\epsilon_q).$$

□

Lemma B.7. *In an economy with 2 states and 2 agents with diverse beliefs, for any $\epsilon_b > 0$ and $\epsilon_q \in (0,1)$ there exists $\bar{\alpha}_l$, $K < \infty$ and $\delta \in (0,1)$ such that for all $\alpha \in (0, \bar{\alpha}_l)$*

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{p_t^M>\pi^{BCP}-\epsilon_b\}} \geq 1-\epsilon_q\right\} \geq 1-K\delta^T$$

Proof. Define $Z_t = -Y_t$ and repeat the steps of the previous lemma. □

Theorem B.2. *In an economy with 2 agents, for any $\epsilon > 0$ there exists $\bar{\alpha}$ such that for all $\alpha \in (0, \bar{\alpha})$*

$$\bar{d}(P||p^M_\alpha) < d(P||\pi^{BCP}) + \epsilon, \ \ P\text{-a.s.}.$$

Proof. We consider two cases. First: $\pi^{BCP} = \pi^{BIP}$, and second, $\pi^{BCP} \neq \pi^{BIP}$.

- First:c$\pi^{BCP} = \pi^{BIP} \Rightarrow^{\text{by Prop.3.3}} \bar{d}(P||\pi^{BCP}) = \bar{d}(P||p^M_\alpha) = \bar{d}(P||\pi^{BIP}), \forall\alpha$ P-a.s..
- Second: $\pi^{BCP} \neq \pi^{BIP}$.

By Lemma B.6 and Lemma B.7, given an $\epsilon_b > 0$ and $\epsilon_q \in (0,1)$ there exists an $\bar{\alpha} = \min\{\bar{\alpha}_l, \bar{\alpha}_r\}$ such that for all $\alpha \in (0, \bar{\alpha})$

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{|p_t^M - \pi^{BCP}| < \epsilon_b\}} \geq 1 - \epsilon_q\right\} \geq 1 - K\delta^T.$$

By continuity of the K-L divergence, for all $\epsilon_d > 0$ there exists an ϵ_b such that $|p_t^M - \pi^{BCP}| < \epsilon_b$ implies $|d(P||p_t^M) - d(P||\pi^{BCP})| < \epsilon_d$. Thus

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| < \epsilon_d\}} \geq 1 - \epsilon_q\right\} \geq 1 - K\delta^T\,; \tag{B.5}$$

and, equivalently,

$$P\left\{\frac{1}{T}\sum_{t=1}^{T} I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| \geq \epsilon_d\}} \leq \epsilon_q\right\} \geq 1 - K\delta^T. \tag{B.6}$$

Equations (B.5) and (B.6) imply that, $\forall T$,

$$\begin{aligned}
&P\left\{\frac{\sum_{t=1}^T d(P||p_t^M) - d(P||\pi^{BCP})}{T} < \epsilon\right\} \\
=&P\left\{\frac{\sum_{t=1}^T d(P||p_t^M) - d(P||\pi^{BCP}) I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| < \epsilon_d\}}}{T} + \frac{\sum_{t=1}^T d(P||p_t^M) - d(P||\pi^{BCP}) I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| \geq \epsilon_d\}}}{T} < \epsilon\right\} \\
\geq&P\left\{\epsilon_d \frac{\sum_{t=1}^T I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| < \epsilon_d\}}}{T} + \max_{p^M \in [\pi^l, \pi^r]}\{d(P||p^M) - d(P||\pi^{BCP})\} \frac{\sum_{t=1}^T I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| \geq \epsilon_d\}}}{T} < \epsilon\right\} \\
\geq^a&P\left\{\max_{p^M \in [\pi^l, \pi^r]}\{d(P||p^M) - d(P||\pi^{BCP})\} \frac{\sum_{t=1}^T I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| \geq \epsilon_d\}}}{T} < \epsilon - \epsilon_d\right\} \\
=^b&P\left\{\frac{\sum_{t=1}^T I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| \geq \epsilon_d\}}}{T} < \frac{\epsilon - \epsilon_d}{\max_{p^M \in [\pi^l, \pi^r]}\{d(P||p^M) - d(P||\pi^{BCP})\}}\right\} \\
\geq&1 - K\delta^T.
\end{aligned}$$

(a): Because $\frac{\sum_{t=1}^T I_{\{|d(P||p_t^M) - d(P||\pi^{BCP})| < \epsilon_d\}}}{T} \leq 1$;
(b): by Equation (B.6), with $\epsilon_q < \frac{\epsilon - \epsilon_d}{\max_{p^M \in [\pi^l, \pi^r]}\{d(P||p^M) - d(P||\pi^{BCP})\}}$.

Where ϵ_d, ϵ_q depends on α but are independent of T.

Let $F_T := \left\{\frac{\sum_{t=1}^T d(P||p_t^M) - d(P||\pi^{BCP})}{T} < \epsilon\right\}$.
The only thing left to show is that $P\left\{\lim_{T\to\infty} F_T\right\} = 1$:

$$P\left\{\lim_{T\to\infty} F_T\right\} \geq P\left\{\liminf_{T\to\infty} F_T\right\} = 1 - P\left\{(\liminf_{T\to\infty} F_T)^C\right\} = 1 - P\left\{\limsup_{T\to\infty} F_T^C\right\} =^a 1$$

(a): by Borel-Cantelli lemma: $\lim_{T\to\infty} \sum_{t=1}^{T} P\left\{F_t^C\right\} < \infty \Rightarrow P\left\{\limsup_{T\to\infty} F_T^C\right\} = 0$;
which, in our context holds because $\lim_{T\to\infty} \sum_{t=1}^{T} P\left\{F_t^C\right\} \leq \lim \sum_{t=1}^{T} K\delta^t < \infty$.

□

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