Do not follow a weak leader

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Abstract

What does it take to survive in the market? Previous literature has proposed (only) sufficient conditions for the consumption share of a trader to converge to 0 (i.e., vanish), which depend on pairwise comparisons of traders’ discounted beliefs. We propose an alternative condition for a trader to vanish that is both necessary and sufficient that depends on the comparison between trader’s discounted beliefs and equilibrium prices. We apply our condition to study the performance of the simple “follow the leader strategy”, which prescribes a trader to mimic the beliefs of the trader who did best in the past. Our condition shows that this strategy leads to certain ruin whenever there is not a unique leader; a result that cannot be obtained using the existing approach.

JEL Classification: D51, D01, G1

1 Introduction

More than a half-century ago economists hypothesized that traders with poor forecasting abilities progressively lose wealth against traders with more accurate probabilistic views (market selection hypothesis, Alchian (1950) and Friedman (1953)). For the general equilibrium settings with complete markets and bounded aggregate endowment, previous literature has formalized this hypothesis and found sufficient conditions for a trader to vanish based
on the pairwise comparison of traders’ accuracy (and discount factors) (Sandroni (2000), Blume and Easley (2006), Yan (2008), Borovička (2013), Massari (2015a)). In a nutshell, these conditions tell us that a trader vanishes whenever there is another trader who is more accurate. This approach greatly simplifies the analysis of heterogeneous-belief economies as it can be applied without solving for the competitive equilibrium (traders’ discount factors and beliefs are exogenous). However, it overlooks a fundamental aspect of competition: in financial markets, traders interact with several traders simultaneously, not in a pairwise fashion. Consequently, the pairwise comparisons approach cannot deliver a condition for a trader to vanish that is both necessary and sufficient (see Blume and Easley (2009)).

To account for this characteristic of financial markets, we propose an alternative approach that is closer to the actual trading experience. We compare traders’ investment decisions against equilibrium prices. This comparison preserves the one-against-many nature of market interactions (prices are equivalent to a convex combination of traders’ beliefs, see Section 4.1) and delivers a condition that is both necessary and sufficient for a trader to vanish.

The main difficulty with our condition is its dependence on equilibrium prices, an endogenous quantity. Fortunately, we do not need to know equilibrium prices exactly to discuss traders’ survival: an approximation of the rate at which they converge to 0 will suffice. Under mild assumptions, we show that equilibrium prices can be well-approximated by a convex combination of traders’ discounted beliefs. Since traders’ beliefs and discount factors are exogenous, our condition maintains the simplicity of the previous approach (can be applied without solving for equilibrium) yet capturing the one-against-many aspect of competition.

For the case in which all traders have the same discount factor, our approximation proves the long-standing conjecture of Blume and Easley (1993) that, irrespective of traders’ risk

\footnote{For the specific case in which traders’ beliefs and the data generating process are iid, Blume and Easley (2009) already proposed a (almost) necessary and sufficient condition for a trader to vanish. Their condition relies on an elegant geometric construction that compares each trader’s discounted beliefs against the beliefs of all the other traders at once. Its main shortcoming is that it cannot be generalized beyond the iid setting (although most investment strategies are not iid in reality) and that it differs from the existing one only for knife-edge choices of traders’ discount factor. The main advantage of our condition with respect to theirs is that our characterization does not require traders’ beliefs or the true data generating process to be iid. This level of generality allows to characterize the performance of intuitive investment strategies that could not be previously analyzed.}
attitudes, equilibrium prices (a risk-adjusted average of traders’ beliefs) evolve in a way that is qualitatively Bayesian (a non-risk-adjusted average of probabilities). In this setting, our condition reads as: a trader vanishes if and only if his beliefs are less accurate than the probability obtained via Bayes’ rule from a regular prior on the set of traders’ beliefs.² Whereas the existing conditions read as: a trader vanishes if there is another trader who is more accurate.

In Section 5 we apply our condition to discuss the performance of the “follow the leader strategy”, (henceforth referred to as FLS): the strategy that prescribes mimicking, in every period, the beliefs of the trader who has been the most successful in the past. We find that if leaders change over time, weak leaders, an FLS-trader is doomed to vanish irrespective of the true data generating process and of traders’ beliefs.

The intuition behind the result is straightforward: i) the FLS-trader cannot outperform the current leader (because he is copying him); ii) the FLS-trader starts copying each new leader with a small delay (because the new leader must first outperform the previous leader); iii) during these delays, the FLS-trader does worse than the new leader (because he is still following the previous leader); and iv), the FLS-trader’s total losses with respect to every new leader increase linearly in the number of previous leaders (because delay-induced losses accumulate over time).³ This intuitive argument requires us to compare the performance of a single trader against the performance of more than one trader at once (leaders change over time), thus it is not compatible with the standard pairwise approach.

In Section 5.1 we show that pairwise comparison of traders’ accuracy can fail to correctly characterize the performance of the FLS. We present an example of a three-trader economy (traders 1,2 and an FLS-trader) in which the FLS-trader vanishes almost surely and yet pairwise comparison of traders’ discounted beliefs fails to indicate that he vanishes. The reason is that, even if the FLS-trader’s consumption-share converges to 0, his consumption-

²A prior is regular if it attaches strictly positive probability to every probability in its prior support.
³This argument holds in a much more general setting than the one we adopt in this paper. For example, it can be applied almost verbatim to any real-life investment decision involving N assets. By the same logic illustrated above, if we adjust our portfolio dynamically to solely hold the asset with the highest past growth rate (leading asset) and if the leading assets changes over time, then we obtain an accumulated return lower than what we would have had by simply holding a long diversified portfolio.
share is higher than that of any other trader infinitely often (because the leader and the follower alternates and the follower loses money at a faster rate than the FLS-trader does). The existing conditions for a trader to vanish are either inapplicable (Th.2 and 3, Blume and Easley (2009)); fail to indicate that the FLS-trader vanishes (Prop.3, Sandroni (2000)); or incorrectly indicate that traders 1 and 2 vanish (Th.8, Blume and Easley (2006)).

2 The model

2.1 The environment

The model is an infinite horizon Arrow-Debreu exchange economy with complete markets. Time is discrete and begins at date 0. At each date, the economy can be in \( S \) mutually exclusive states: \( S := \{1, \ldots, S\} \), with cartesian product \( S^t = S \times S \). The set of all infinite sequences of states, paths, is \( \Sigma := \times^\infty S \), with representative path, \( \sigma = (\sigma_1, \ldots) \). \( \sigma^t = (\sigma_1, \ldots, \sigma_t) \) denotes the partial history till period \( t \), \( C(\sigma^t) \) the cylinder set with base \( \sigma^t \), \( C(\sigma^t) = \{ \sigma \in \Sigma | \sigma = (\sigma^t, \ldots) \} \), \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by the cylinders, \( \mathcal{F}_t = \sigma (C(\sigma^t), \forall \sigma^t \in S^t) \), and \( \mathcal{F} \) the \( \sigma \)-algebra generated by their union, \( \mathcal{F} = \sigma (\bigcup_t \mathcal{F}_t) \). By construction \( \{ \mathcal{F}_t \} \) is a filtration. In what follows we introduce a number of economic variables, which depends on \( \sigma^t \), these variables are assumed to be date \( t \) measurable according to the natural \( \mathcal{F}_t \).

For any probability measure \( p \) on \( \Sigma \), \( p(\sigma^t) := p(\{ \sigma_1 \times \ldots \times \sigma_t \} \times S \times S \times \ldots) \) is the marginal probability of the partial history \( \sigma^t \) while \( p(\sigma^t|\sigma^{t-1}) = \frac{p(\sigma^t)}{p(\sigma^{t-1})} \) is the conditional probability of the last observation of the partial history \( \sigma^t \) given its first \( t-1 \) realizations.

2.2 Traders

The economy contains a finite set of traders \( I \). Each trader, \( i \), has consumption set \( \mathbb{R}^{++}_+ \). A consumption plan \( c : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}^{++}_+ \) is a sequence of \( \mathbb{R}^{++}_+ \)-valued functions \( \{ c_t(\sigma) \}_{t=0}^{\infty} \). Each trader \( i \) is characterized by a payoff function \( u^i : \mathbb{R}^{++}_+ \to \mathbb{R} \) over consumption, a discount

\[4\] The example highlights another case (see Massari (2013)) in which Th. 8 leads to an incorrect conclusion.

\[5\] For the sake of notation, we are assuming that past realizations constitute all of the relevant information, i.e. \( \mathcal{F}_t := \sigma^t \).
factor $\beta_i \in (0, 1)$, an endowment stream $\{e^i_t(\sigma)\}_{t=0}^{\infty}$ and a subjective probability $p^i$ on $\Sigma$, his beliefs. $\mathcal{P} = \{p^i : i \in \mathcal{I}\}$ is the set of traders’ beliefs. Trader $i$’s utility for a consumption plan $c$ is:

$$U^i(c) = E_{p^i} \sum_{t=0}^{\infty} \beta^i_t u_i(c_t(\sigma)).$$

We adopt the nomenclature from the selection literature, where traders can either vanish or survive.

**Definition 1.** Trader $i$ vanishes on path $\sigma$ if $\lim_{t \to \infty} c_t(\sigma) = 0$. He survives on path $\sigma$ if $\limsup c_t(\sigma) > 0$.

Finally, we rank the accuracy of beliefs according to their likelihood:

**Definition 2.** Given a true probability $P$, $p^i$ is more accurate than $p^j$ if $p^j(\sigma_t) \rightarrow p^i(\sigma_t)$ $P$-a.s. 0.

This exact criterion for accuracy improves on the existing one (Sandroni (2000) and Blume and Easley (2006)) which only approximate traders’ likelihoods (see Appendix A). We decided for this criterion to ensure that the inability of the pairwise comparison approach to provide a condition for a trader to vanish that is both necessary and sufficient cannot be attributed to an approximation error.

### 2.3 Competitive Equilibrium

We derive our results using the time 0 trading setting. $q(\sigma^t)$ denotes the date 0 price of a claim that pays a unit of consumption at the end of $\sigma^t$ in terms of consumption at time 0. A competitive equilibrium is a sequence of prices and, for each trader, a consumption plan that is affordable, preference maximal on the budget set and mutually feasible. Sufficient assumptions for the existence of the competitive equilibrium are (Peleg and Yaari (1970)):

- **A1:** The payoff functions $u_i : \mathbb{R}_+ \to [-\infty, +\infty]$ are $C^1$, strictly concave, increasing and satisfy the Inada condition at 0; that is, $u'_i(c) \to \infty$ as $c \searrow 0$.

- **A2:** There are numbers $0 < f \leq F < +\infty$ such that for each trader $i$, all dates $t$ and all paths $\sigma$, $f \leq \inf_{\sigma^t} \sum_{i \in \mathcal{I}} e^i_t(\sigma) \leq \sup_{\sigma^t} \sum_{i \in \mathcal{I}} e^i_t(\sigma) \leq F$. 

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• A3: For all traders $i$, all dates $t$ and all paths $\sigma$, $p^i(\sigma^t) > 0 \Leftrightarrow P(\sigma^t) > 0$.

Assumption A1 is a collection of standard properties for the payoff functions. Assumption A2 uniformly bounds the aggregate endowment above and away from 0. This assumption is standard in the selection literature although not necessary for the existence of the competitive equilibrium. We maintain this assumption to obtain a nice characterization of asymptotic equilibrium prices. Assumption A3 rules out pathological cases of non-existence of the competitive equilibrium due to traders’ disagreement on 0 probability events.

3 Comparing the two approaches

In this Section, we make use of an example to illustrate the main features of our approach and compare it with the existing one. Our approach is shown to be more informative (we directly characterize equilibrium prices) and more precise (our condition is both necessary and sufficient for a trader to vanish) than the existing one. Its main drawback is its dependence on equilibrium prices, an endogenous quantity. This problem is addressed and solved in the next Section, in which we provide a simple characterization of equilibrium prices that applies to every economy that satisfies A1-A3.

Example 1: Consider an Arrow’s security economy with $S$ states with iid multinomial distribution $P$ and constant aggregate endowment: $\forall t, \forall \sigma, \sum_{i \in \mathcal{I}} e^t_i(\sigma) = 1$. There are $\mathcal{I}$ traders with log utility, identical discount factors $(\beta)$, iid beliefs $p^i$ and positive Pareto weight, $\frac{1}{c^i_0}$. Every trader in the economy aims to solve:

$$\max_{\{\sigma^t(\sigma)\}} \mathbb{E}_{\rho^i} \sum_{t=0}^{\infty} \beta^t \ln(c_i^t(\sigma)) \quad s.t. \quad \sum_{t=0}^{\infty} \sum_{\sigma^t \in \Sigma^t} q^t(\sigma) (c_i^t(\sigma) - c_i^t(\sigma)) \leq 0.$$
Traders’ first-order conditions of the maximization problem are sufficient for the pareto optimum and, in every path $\sigma^t$, can be expressed as

$$\frac{\beta^t p^i(\sigma^t)}{c^i(\sigma)} = \frac{q(\sigma^t)}{c^0}.$$

Our approach:

- **First step**: to characterize equilibrium prices.

  In this simple economy, equilibrium prices can be obtained explicitly from the FOC:

  $$\forall t \forall \sigma \in \Sigma, \quad q(\sigma^t) = \beta^t \sum_{i \in I} p^i(\sigma^t) c^i_0$$

**Remark**: As previously noted (Rubinstein (1974), Blume and Easley (2009), Massari (2015a)), in a log economy with homogeneous discount factors, equilibrium prices coincide with the discounted probabilities $(p^B(\sigma^t))$ obtained via Bayes’ rule from a prior distribution $C_0 = \{c^1_0, ..., c^I_0\}$ on $P = \{p^i : i \in I\}$: $q(\sigma^t) = \beta^t \sum_{i \in I} p^i(\sigma^t) c^i_0 := \beta^t p^B(\sigma^t)$.

- **Second step**: to use equilibrium prices to discuss survival.

  Substituting the price equation in the FOC: $c^i_t(\sigma) = \beta^t \frac{p^i(\sigma^t)c^i_0}{q(\sigma^t)} = \frac{p^i(\sigma^t)c^i_0}{p^B(\sigma^t)}$.

  Thus, $c^i_t(\sigma) \to P\text{-a.s.} 0 \Leftrightarrow \frac{p^i(\sigma^t)c^i_0}{p^B(\sigma^t)} \to P\text{-a.s.} 0$: trader $i$ vanishes $P$-a.s. if and only if his beliefs are less accurate than the probability obtained via Bayes’ rule from $C_0$.

The existing approach:

The existing approach skips the characterization of equilibrium prices and directly focuses on pairwise comparison of traders’ discounted beliefs. Taking the log ratios of the FOCs of different traders, prices simplify and we obtain:

$$\log \frac{c^i_t(\sigma)}{c^i_1(\sigma)} = \log \frac{p^i(\sigma^t)c^j_0}{p^j(\sigma^t)c^i_0}.$$  \hspace{1cm} (1)

By A2, the aggregate endowment is bounded, thus $\log \frac{p^i(\sigma^t)c^j_0}{p^j(\sigma^t)c^i_0} \to P\text{-a.s.} -\infty \Rightarrow c^i_t(\sigma) \to P\text{-a.s.} 0$: trader $j$ vanishes $P$-a.s. if there is another trader whose beliefs are more accurate.

Approximating the right hand side of Eq.1 with the difference of traders’ Entropy (See Appendix A) we obtain Sandroni (2000)’s condition (Prop.3), while approximating it with the difference of traders’ sum of expected relative entropy (See Appendix A) we obtain Blume
and Easley (2006)’s (Th.8).

4 Main result

In this Section we characterize equilibrium prices and construct a necessary and sufficient condition for a trader to vanish. Our condition is then applied to the study of the performance of the FLS and to show that the pairwise comparison approach is too coarse to analyze this simple strategy.

4.1 Equilibrium prices

Example 1 highlights that, for log-economies with homogeneous discount factors, equilibrium prices coincide with the discounted probabilities obtained via Bayes’ rule from a prior $C_0$ on the set of trader’s beliefs $\mathcal{P}$. The generalization of this result is potentially complicated by two factors. First, fluctuations of the aggregate endowment alter investment decisions. Second, risk attitudes affect wealth-shares’ convergence rates (Blume and Easley (2006), Massari (2015a)): if traders are less risk averse, they trade more aggressively and wealth-shares move faster. Theorem 1 shows that both effects are asymptotically negligible: equilibrium prices are asymptotically equivalent to a convex combination of traders’ discounted beliefs.

Theorem 1. In an economy that satisfies A1-A3, $\forall \sigma \in \Sigma$:

$$q(\sigma) \simeq \sum_{i \in I} \beta_t^i p^i(\sigma^t) c_0^i.$$  

Where $\simeq$ captures the notion of two functions that converge to 0 at the same rate\(^8\)

Proof. See Appendix B.

This approximation is valid on every path, thus independent of the true probability, and for every finite set of beliefs $\mathcal{P}$, i.e. traders’ beliefs need not be iid. In other words, Theorem

\(^8\)Definition: Given two functions $f(x)$ and $g(x)$, $f(x) \simeq g(x)$ if and only if $\left\{ \begin{array}{l} \limsup \frac{f(x)}{g(x)} < \infty \\ \liminf \frac{f(x)}{g(x)} > 0 \end{array} \right.$
1 allows us to generalize the argument of the previous example from the log utility/iid beliefs setting to all economies that satisfy A1-A3. For the specific case in which all traders have the same discount factor, Theorem 1 proves the long-standing conjecture of Blume and Easley (1993) that equilibrium prices (a risk-adjusted average of traders’ beliefs) evolve in a way that is qualitatively Bayesian (a non-risk-adjusted average of probabilities):

**Corollary 1.** In an homogeneous-discount-factors-economy that satisfies A1-A3, equilibrium prices are mutually absolutely continuous with the discounted probabilities obtained via Bayes’ rule from a regular prior, $g$, on $\mathcal{P}$

**Proof.**

\[
\forall \sigma, \quad \frac{q(\sigma)}{p^B(\sigma)} \sim_{By \; Th.1} \frac{\beta^t \sum_{i\in I} p^i(\sigma^t) c^i_0}{\beta^t \sum_{i\in I} p^i(\sigma^t) g^i} \sim 1
\]

\[\square\]

### 4.2 Necessary and Sufficient condition to vanish

Our approach addresses the problem of wealth-share convergence by directly comparing traders’ discounted beliefs and equilibrium prices. Our condition tells us that a trader vanishes on those histories in which he subjectively believes that consumption costs too much.

**Theorem 2.** In an economy that satisfies A1-A3,

\[i) \quad \limsup \frac{\beta^t p^i(\sigma^t) c^i_0}{q(\sigma^t)} = 0 \iff \text{trader } i \text{ vanishes on path } \sigma\]

\[ii) \quad E\left(\limsup \frac{\beta^t p^i(\sigma^t) c^i_0}{q(\sigma^t)}\right) = 0 \iff \text{trader } i \text{ vanishes } P\text{-a.s.}\]

**Proof.** See Appendix B. \[\square\]

The main difficulty with this condition is its dependence on equilibrium prices, an endogenous quantity. Under A1-A3, Theorem 1 provides the needed characterization. For homogeneous discount factors economies, it allows to apply known consistency results in Bayesian
statistics to our selection problem. Since traders’ beliefs are exogenous and Bayesian inference is well understood, our condition is, in most cases, easy to verify.

**Corollary 2.** In a homogeneous-discount-factor-economy that satisfies A1-A3, a trader vanishes $P$-a.s. if and only if his beliefs are less accurate than the beliefs of a Bayesian with regular prior on $\mathcal{P}$.

**Proof.** It follows from Th. 2 and Cor. 1.

Consistent with the existing results, our condition implies that risk attitudes do not affect survival. More specifically, Theorem 2 shows that risk attitudes can affect survival only by affecting the rate at which equilibrium prices converge to 0. While Theorem 1 shows that, under A1-A3, risk attitudes have no effect on this rate.

**5 Do not follow a weak leader**

We are now ready to apply our condition to discuss the performance of the follow the leader strategy: the strategy that prescribes mimicking the investment strategy of the most successful trader in the economy. We find that this strategy performs well if and only if a unique most successful trader is present in the market. In every path in which there are at least two leaders who alternate infinitely often, a follower of the FLS is doomed to ruin.

**Proposition 1.** In homogeneous-discount-factor-economy that satisfies A1-A3, if there are at least two traders that alternate in their leadership, a trader that follows the FLS vanishes.

**Proof.** See Appendix B.

This result does not rely on specific assumptions about the data generating process, or on the strategy adopted by the other traders. It applies to every situation in which the data generating process or the investment strategy of the traders are such that the leader changes infinitely often. Given that successful investors in financial markets tend to alternate over time, our result strongly suggest to avoid the FLS. The normative aspect of our recommendation is even stronger, given that our result is derived in a setting in which there are no
switching costs and in which the FLS-trader knows exactly the leaders’ beliefs.

An intuition for the result is the following. Suppose you are driving in heavy traffic. There are two lines of cars (1,2), WLOG you begin in line 1. Denote the car in front of you by car 1 and the car next to it by car 2. Your goal is to arrive not much later\(^9\) than the fastest car between car 1 and car 2 (the leader’s car). The follow the leader strategy is, loosely speaking, equivalent to the strategy that prescribes to always be in the line of the car that is ahead (follow the leader’s car). This strategy can be summarized as follows.\(^{10}\)

- At \(t=0\), you are in line 1, behind car 1, and car 1 and car 2 are next to each other. If car 1 moves one position ahead of car 2, you stay in line 1 (behind car 1) and remain in line 1 until car 1 and car 2 are next to each other again, \(t^*_1\). If car 2 moves one position ahead of car 1, you switch line from 1 to 2, (now you have one car between your car and car 2) and remain in line 2 until car 1 and car 2 are next to each other again (and one car ahead of you), \(t^*_2\).

- At \(t^*_1\) you repeat the strategy. If car 1 was faster, nothing has changed. If car 2 was faster, the strategy remains the same, but now you start in line 2 and you are one car behind. By the previous argument, if car 2 remains ahead, you stick to line 2 (one car behind car 2), while if car 1 moves ahead, you switch to line 1 and lose again one position, you are now two cars behind car 1.

- Iteratively, you lose one position against the leader’s car every time a change in leadership occurs. Thus, if the two lines are equally bad (the leader changes infinitely often), the FLS dooms you to lag increasingly further behind the leader’s car.

Example 2 illustrates the result when the sequence of realized states is deterministic.

**Example 2** Consider an economy with two states \(S = \{a, b\}\) and three traders (1,2,FLS-trader) with identical Pareto weights \((\frac{1}{3})\), discount factors \(\beta\) and log utility. Traders 1 and

\(^9\)In our economy, the FLS-trader’s wealth-share is mostly affected by the leader’s wealth.

\(^{10}\)WLOG, we assume that when the two cars are next to each other you remain do not change line.
2 have fixed beliefs: \( p^1(a) = p^2(b) = \frac{1}{3} \). The FLS-trader always adopts the beliefs of the trader that did best in the past and uses \( \frac{1}{2} \) when trader 1 and 2 did equally well. The sequence of states is deterministic: \( \{a, b, a, b, a....\} \). In this simple setting, the FLS prescribes to copy trader 2 in odd periods and to use \( \frac{1}{2} \) in even periods. A quick calculation shows that
\[
\lim_{t\to\infty} \frac{c_{FLS}(\sigma)}{q(\sigma^t)} = \lim_{t\to\infty} \frac{\frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2}{\frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2} = 0.
\]

### 5.1 Discussion

In this section we show a case in which the standard approach, ignoring the effect of group competition, is unable to correctly analyze the performance of the FLS. We provide an example in which, even adopting our exact measure of accuracy, pairwise comparison of individual beliefs must fail to correctly characterize the performance of a trader who follows the FLS. In the example, there are three traders, all traders satisfy the survival criterion according to our precise notion of accuracy (Equation 1), and yet the FLS-trader vanishes P-a.s.. A quick calculation shows that no trader in the economy satisfies Sandroni’s condition to vanish and that Blume-Easleys’ condition incorrectly indicates that the trader that follows the FLS dominates.

**Example 3:** Consider an economy with two states \( S = \{a, b\} \), iid true probabilities \( P = [\frac{1}{2}, \frac{1}{2}] \) and three traders (1,2,FLS) with identical Pareto weights \( \frac{1}{c_0} \), discount factors \( \beta \) and log utility. Traders 1 and 2 have fixed beliefs: \( p^1(a) = p^2(b) = \frac{1}{3} \). The FLS-trader copies the beliefs of the trader that did best in the past in those periods in which there is a leader and gives the two strategies equal weight otherwise:

\[
p^{FLS}(a_t|\sigma^{t-1}) = \begin{cases} p^i(a), & i : p^i(\sigma^{t-1}) = \arg\max\{p^1(\sigma^{t-1}), p^2(\sigma^{t-1})\} \\ \frac{1}{2}, & \text{if ties occur.} \end{cases}
\]

\[11\]With this tie-breaking rule, the FLS coincides with the SNML algorithm (Roos and Rissanen (2008)) and with the Bayesian prediction from \( C_0 \), under the 0-1 loss function. In the example, this tie-breaking rule is the most profitable for the FLS-trader: \( \frac{1}{2} = P \). Thus the result applies to any tie-breaking rule.
Proposition 2. The FLS-trader vanishes $P$-a.s.:

$$\limsup \ c^3(\sigma^t) = \limsup \frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = 0 \quad P$-a.s.$$

Proof. Appendix C

Proposition 2 is a special case of Proposition 1. It follows by noticing that trader 1 and 2 alternate infinitely often in their leadership because their beliefs are equally accurate.

Proposition 3. Pairwise comparison does not imply that the FLS-trader vanishes:

$$\text{for } i = 1, 2: \limsup \frac{c^{FLS}(\sigma)}{c^i(\sigma)} = \limsup \frac{p^{FLS}(\sigma^t)}{p^i(\sigma^t)} = \infty \quad P$-a.s.$$

Proof. Appendix C

The intuition is as follows: WLOG, consider the comparison between the FLS-trader and trader 1. The FLS-trader’s strategy “loses ground” every time there is a change in leadership, thus the FLS-trader becomes further behind trader 1 when trader 1 is the leader. However, when trader 2 is the leader, the FLS-trader is doing better than trader 1. The result follows verifying that the accumulated gains on these deviations are infinitely often superior than the accumulated losses that the FLS-trader incurs because of leader’s alternation.

Continuing with our car intuition, Proposition 3 tells us that (if each car has 50% chance to move one position ahead in every period) you will be infinitely often ahead of the follower’s car. This result is certainly correct. However, we are not interested about your performance against the follower’s car. We want to know how close you are driving to the leader’s car! In economic terms, if the consumption-share of the follower becomes arbitrarily close to 0, knowing that you have higher consumption than the follower does not tell us much about your actual consumption-share.

Comparing the two propositions, we conclude that even if the FLS-trader is not fit to
survive against both trader 1 and 2 simultaneously,\footnote{The deviations of the empirical average from the true probability that favors him against trader 1 are the same deviations that cause him to perform poorly against trader 2 and vice versa.} he would be fit to survive against traders 1 or trader 2 in isolation. Thus pairwise comparison of individual characteristics cannot deliver a necessary and sufficient condition for a trader to vanish.

Remarks
Not surprisingly, the existing conditions to vanish, an approximation of what we use in Prop.3, fail to correctly characterize the performance of the FLS strategy. In particular,

- no trader satisfies Sandroni (2000)’s condition to vanish. The average beliefs of all traders are equally accurate because the likelihood ratio diverges at a rate slower than $t$: $\ln \frac{p^{FLS}(\sigma_t)}{p_1(\sigma_t)} \sim \sqrt{t} \Rightarrow \lim_{t \to \infty} \frac{1}{t} \ln \frac{p^{FLS}(\sigma_t)}{p_1(\sigma_t)} = \lim_{t \to \infty} E_{3,t} - E_{1,t} = 0$;

- traders 1 and 2 satisfy Blume and Easley (2006)’s condition to vanish:

$$\lim_{t \to \infty} \sum_{\tau=1}^{t} E_P \ln \frac{p^{FLS}(\sigma_\tau|\sigma_{\tau-1})}{P(\sigma_\tau)} - \sum_{\tau=1}^{t} E_P \ln \frac{p_1(\sigma_\tau|\sigma_{\tau-1})}{P(\sigma_\tau)} = +\infty. \footnote{The result follows by noticing that the FLS-trader is more accurate than trader 1 and 2 infinitely often (whenever trader 1 and trader 2 are equally wealthy he uses the correct model: $p^{FLS}(\sigma_{t+1}|\sigma^*) = \frac{1}{2}$) and never less accurate.}$$

Its application would lead us to the incorrect conclusion that the FLS-trader dominates. The example highlights another case (different from the one discussed in Massari (2013)) in which the simple comparison of the accuracy of one period ahead forecasts gives misleading indications of the accuracy of forecasts on sequences of realizations.

6 Conclusions
This paper continues the project started by Sandroni (2000) and Blume and Easley (1992) on market selection. Unlike the existing approaches, we focus on the role played by equilibrium prices in the selection mechanism. We show that equilibrium prices are asymptotically equivalent to a convex combination of traders’ discounted beliefs. Hence, if all traders have the same discount factor, equilibrium prices are asymptotically equivalent (mutually...
absolutely continuous) to the discounted probability obtained via Bayes’ rule from a regular prior on the set of traders’ beliefs. Our characterization of equilibrium prices is used to derive a condition that is necessary, as well as sufficient, for a trader to vanish. We use our condition to study the performance of the FLS. We find that, irrespective of the true probability, the FLS does not guarantee survival whenever there is not a unique leader in the economy. We show that because the existing approach overlooks the one-against-many aspect of competition in financial markets, it is not well-suited to study these types of strategies. Similar limitations are likely to have a bite whenever traders use strategies that depend on other traders’ behavior or prices. In these situations, we recommend abandoning the pairwise comparison approach in favor of our condition.

A Appendix

In this Appendix we compare our notion of accuracy with Sandroni (2000)’s and Blume and Easley (2006)’s. Our notion of accuracy ranks traders’ beliefs according to their likelihood:

**Definition 3.** Given a true probability $P$, $p^i$ is more accurate than $p^j$ if $\frac{P(\sigma^t)}{p^i(\sigma^t)} \to P$ a.s. 0.

Our criterion is more precise than the previously adopted (Sandroni (2000) and Blume and Easley (2006)) which only approximate traders’ likelihoods, and it relates to them as follows:

**Sandroni (2000). Entropy of trader $i$:**

$$E_{i,t} := -\frac{1}{t} \sum_{\tau=1}^{t} E_P \left( \ln \frac{P(\sigma_\tau | \sigma_{\tau-1})}{p^i(\sigma_\tau | \sigma_{\tau-1})} \right)$$

(2)

If we consider $E_P \left( \ln \frac{P(\sigma_\tau | \sigma_{\tau-1})}{p^i(\sigma_\tau | \sigma_{\tau-1})} \right)$ to be a measure of trader’s expected accuracy, the comparison of traders’ entropies is a comparison of the average expected accuracy of two traders. As intuition suggests, $E_{i,t} - E_{j,t}$ is a coarse approximation of trader’s average likelihood ratio: if trader $i$’s expected beliefs are on average more accurate than trader $j$’s, then trader $i$ is empirically more accurate than trader $j$ (as an application of the Strong Law of Large Numbers for Martingale Differences, see Sandroni (2000)). However, the converse implication does not always follow because the averaging term, $\frac{1}{t}$, “kills” divergence rates that are slower than $t$. For example, it cannot distinguish between the different learning rates of two Bayesian with different prior support dimensionality (see Blume and Easley (2006)).
Blume-Easley (2006). Sum of conditional relative entropies of trader $i$:

$$D_t(P||p_i) := \sum_{\tau=1}^{t} E_P \left( \ln \frac{P(\sigma_\tau||\sigma_{\tau-1})}{p_i(\sigma_\tau||\sigma_{\tau-1})} \right) \quad (3)$$

This definition suggests that if trader $i$ is in every period, on expectation, more accurate than trader $j$ he should also be empirically more accurate. This intuitive argument is, however, not always correct: there are cases in which $D_t(P||p_i) - D_t(P||p_j) \to \infty$ even though $\frac{p'(\sigma_t)}{p(\sigma_t)} < \infty$ (Massari (2013)). In these cases we cannot rely on this measure to discuss survival: it leads to incorrect conclusions. In Section 5.1, we present a new case in which the use of this criterion leads to an incorrect conclusion (see remark 2).

The comparison between Equations 2 and 3 highlights that Blume and Easley (2006)'s criterion can lead to incorrect results only if the log likelihood divergence between two traders is $o(t)$. Otherwise (i.e. if the log likelihood divergence rate is $O(t)$), this criterion becomes equivalent to Sandroni (2000)'s one, which is a correct sufficient condition for a trader to vanish.

B Appendix

Lemma 1. In an economy that satisfies A1-A3: $\exists a, b \in (0, \infty) : \forall t, \forall \sigma \in \Sigma$:

$$a < \sum_{i \in I} \frac{1}{u'_i(c^i_t(\sigma))} < b \quad (4)$$

Proof.

i) $\sum_{i \in I} \frac{1}{u'_i(c^i_t(\sigma))} < \infty$: $\forall \sigma \in \Sigma, \forall i \in I, u'_i(c^i_t(\sigma)) > 0$ because the total endowment is finite (A2) and the payoff functions are monotone and strictly concave with positive derivative at 0 (A1).

ii) $0 < \sum_{i \in I} \frac{1}{u'_i(c^i_t(\sigma))}$: by contradiction.

$\sum_{i \in I} \frac{1}{u'_i(c^i_t(\sigma))} = 0 \iff \forall i \in I, u'_i(c^i_t(\sigma)) = \infty$ which is true if and only if all the traders have 0 consumption and satisfy the Inada condition at 0. The first requirements is impossible as it violates the market clearing condition: $\sum_{i \in I} c^i_t = \sum_{i \in I} e^i_t > 0$.

Proof of Theorem 1

Proof. By the FOCs: $\forall i \in I, \forall t, \forall \sigma \in \Sigma, \frac{q(\sigma^t)}{p(\sigma^t)c^i_0} = \beta' u'_i(c^i_t)$.

Rearranging and summing over traders: $q(\sigma^t) = \sum_{i \in I} \frac{\beta'_i p(\sigma^t)c^i_0}{u'_i(c^i_t)} \frac{\sum_{j \in I} \beta'_j p(\sigma^t)c^j_0}{\sum_{j \in I} u'_j(c^j_t)}$.

By Lemma 1 $\exists a, b \in (0, \infty) : \forall t, \forall \sigma \in \Sigma, q(\sigma^t) \in \left[ \frac{\sum_{i \in I} \beta'_i p(\sigma^t)c^i_0}{b}, \frac{\sum_{i \in I} \beta'_i p(\sigma^t)c^i_0}{a} \right]$. Thus $q(\sigma^t) \approx \sum_{i \in I} \beta'_i p(\sigma^t)c^i_0$. 

}\]
Proof of Theorem 2:

Proof.

i) By the FOC: \[ \frac{1}{u_i'(c_i(\sigma))} = \frac{\beta_i^t p^i(\sigma^t)c_i^0}{q(\sigma^t)} \]. Therefore, \( \forall t, \forall \sigma \in \Sigma \),

\[ \limsup \frac{\beta_i^t p^i(\sigma^t)c_i^0}{q(\sigma^t)} = 0 \Leftrightarrow \limsup \frac{1}{u_i'(c_i(\sigma))} = 0 \Leftrightarrow \liminf u_i'(c_i(\sigma)) = +\infty \Leftrightarrow c_i^1(\sigma) \to 0 \]

a: By A1 \( \lim_{c \to 0} u_i'(c) = +\infty \) and the payoff functions are strictly concave, increasing.

ii) \( \forall \sigma, \frac{p_{FLS}^i(\sigma^t) c_i^0}{q(\sigma^t)} \geq 0 \Rightarrow E \left( \limsup \frac{\beta_i^t p^i(\sigma^t)c_i^0}{q(\sigma^t)} \right) = 0 \Leftrightarrow \limsup \frac{\beta_i^t p^i(\sigma^t)c_i^0}{q(\sigma^t)} = 0 \) on a set of measure 1. \( \square \)

Proof of proposition 1

Proof. By Corollary 1 the FLS-trader vanishes if \( \frac{p_{FLS}^i(\sigma^t)}{\sum_{i} p^i(\sigma^t)} \to 0. \)

Let \( \tilde{L}(t) \) be the number of times the leader has changed before \( t \). By assumption, \( \tilde{L}(t) \to \infty. \)

By the same logic used in the proof of Lemma 4 \( \forall \sigma^t, \ln p_{FLS}^i(\sigma^t) = \ln \left( \max_{i \in I} p^i(\sigma^t) \right) - \tilde{L}(t)O(1) \) and the result follows. \( \square \)

C Appendix

Let's focus on the economy of example 3. The generalization of the results to more complex economies follows the same logic but is notationally more complicated.

Our goal is to have a \( P \)-a.s. approximation of \( p_{FLS}^i(\sigma^t) \) and \( q(\sigma^t) \).

Let start by expressing the sequence of realizations as a Random Walk:

\[ y_{\tau} = \begin{cases} 1 & \text{if } \sigma_{\tau} = a \\ -1 & \text{if } \sigma_{\tau} = b \end{cases} \]

and \( S_t = \sum_{\tau=1}^{t} y_{\tau}. \)

We will be interested about two RVs:

- Local Time \( L(t) \): the number of times \( S_t = 0 \) before \( t \): \( L(t) := \sum_{\tau=1}^{t} I_{S_{\tau} = 0} \)
- \( |S_t| \): the absolute distance of the random walk from 0 at \( t \).

The relationship between \( |S_t| \) and \( L(t) \) is captured by Tanaka formula, from which the following lemma follows.

Lemma 2. Let \( S_t \) be a random walk, \( |S_t| \) its absolute value at \( t \) and \( L(t) \) its local time at \( t \), then

\[ \begin{cases} \limsup |S_t| - L(t) = +\infty & P \text{-a.s.} \\ \liminf |S_t| - L(t) = -\infty & P \text{-a.s.} \end{cases} \]
Proof. Define the RV \( z_t = y_t \, \text{sign}(S_{t-1}) \), with \( \text{sign}(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \). By the discrete time version of Tanaka’s formula:

\[
\sum_{\tau=1}^{t} z_{\tau} = |S_t| - L(t) + I_{y_{T_{N+1}=1}}
\]

Because, \( z_t \) are iid RVs with \( E(z_t) = 0 \) and \( \text{Var}(z_t) = 1 \), by the Law of Iterated Logarithms (Williams (1991)):

\[
\begin{align*}
\limsup \frac{\sum_{\tau=1}^{t} z_{\tau}}{\sqrt{2t \log \log t}} &= P \text{-a.s. } 1 \\
\liminf \frac{\sum_{\tau=1}^{t} z_{\tau}}{\sqrt{2t \log \log t}} &= P \text{-a.s. } -1
\end{align*}
\]

Thus, \( \limsup |S_t| - L(t) = P \text{-a.s. } +\infty \) and \( \liminf |S_t| - L(t) = P \text{-a.s. } -\infty \), which proves the lemma.

**Lemma 3.** Let \( S_t \) be a random walk and \( L(t) \) its local time at \( t \), then \( \lim L(t) = +\infty \) P.a.s.

Proof. By construction, \( |S| > 0 \) and \( L(t) > 0 \); by Lemma 2:

\[
\liminf |S_t| - L(t) = -\infty \Rightarrow \lim_{t \to \infty} L(t) = +\infty \text{ P.a.s.}
\]

**Lemma 4.** In the economy of example 3

\[
\forall \sigma \in S^\infty, \quad \ln p^{FLS}(\sigma^t) = \ln \left( \max_{i=1,2} p^i(\sigma^t) \right) - L(t) \ln \frac{4}{3} \quad (5)
\]

Proof. Let \( (\sigma^{t-1}, a) \) be the sequence whose first \( t-1 \) elements coincide with \( \sigma^{t-1} \) and whose last element is \( a \). That is, \( (\sigma^{t-1}, a) := \{\sigma_1, \ldots, \sigma_{t-1}, a\} \). The prediction scheme \( p^{FLS} \) coincides with Sequential Normalized Maximum Likelihood, (Roos and Rissanen (2008), Massari (2015b), Grünwald (2007)) and can be equivalently expressed as:

\[
p^{FLS}(a_t|\sigma^{t-1}) = \frac{\max_i p^i(\sigma^{t-1}, a)}{\max_i p^i(\sigma^{t-1}, a) + \max_i p^i(\sigma^{t-1}, b)} = \frac{\max_{i \in \{1,2\}} p^i(\sigma^{t-1}, a)}{\max_{i \in \{1,2\}} p^i(\sigma^{t-1}, a) + \max_{i \in \{1,2\}} p^i(\sigma^{t-1}, b)}
\]

The denominator satisfies

\[
\frac{\max_{i \in \{1,2\}} p^i(\sigma^{t-1}, a) + \max_{i \in \{1,2\}} p^i(\sigma^{t-1}, b)}{\max_{i \in \{1,2\}} p(\sigma^{t-1})} = \begin{cases} 1 & \text{if } S_{t-1} \neq 0 \\ \frac{4}{3} & \text{if } S_{t-1} = 0 \end{cases}
\]
Thus \( \ln p^{FLS} \) can be equivalently written as (telescoping):

\[
\ln p^{FLS}(\sigma^t) = \sum_{\tau=1}^{t} \ln p^{FLS}(\sigma_{\tau} | \sigma^{\tau-1}) \\
= \sum_{\tau=1}^{t} \ln \frac{\max_{i \in \{1,2\}} p^{i}(\sigma^t)}{\max_{i \in \{1,2\}} p^{i}(\sigma^{\tau-1})} - \sum_{\tau=1}^{t} \ln \frac{\max_{i \in \{1,2\}} p^{i}(\sigma^{\tau-1}, a) + \max_{i \in \{1,2\}} p^{i}(\sigma^{\tau-1}, b)}{\max_{i \in \{1,2\}} p^{i}(\sigma^{\tau-1})} \\
= \ln \max_{i \in \{1,2\}} p^{i}(\sigma^t) - L(t) \ln \frac{4}{3}.
\]

\[\square\]

**Proof of Proposition 2:** \( \lim_{\tau \to \infty} \frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = 0 \) \( P \)-a.s.

Proof. By construction, \( q(\sigma^t) = \frac{1}{3} p^{1}(\sigma^t) + \frac{1}{3} p^{2}(\sigma^t) + \frac{1}{3} p^{FLS}(\sigma^t) \geq \frac{1}{3} \max_{i \in \{1,2\}} p^{i}(\sigma^t) \)

Using the characterization of Lemma 4,

\[
\ln \frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = \ln \frac{\max_{i \in \{1,2\}} p^{i}(\sigma^t)}{\frac{1}{3} p^{1}(\sigma^t) + \frac{1}{3} p^{2}(\sigma^t) + \frac{1}{3} p^{FLS}(\sigma^t)} - L(t) \ln \frac{4}{3} \leq \ln \frac{1}{\frac{4}{3}} - L(t) \ln \frac{4}{3} \rightarrow P \)-a.s. \( -\infty \)

Where the last \( P \)-a.s. convergence follows from Lemma 3

\[\square\]

**Proof of Proposition 3:** For \( i = 1, 2 \), \( \lim \sup_{t \to \infty} \frac{p^{FLS}(\sigma^t)}{p^{i}(\sigma^t)} = +\infty \) \( P \)-a.s.

Proof. WLOG, let focus on \( \lim \sup_{t \to \infty} \frac{p^{FLS}(\sigma^t)}{p^{i}(\sigma^t)} \). Trader’s 1 likelihood is given by:

\[
p^{1}(\sigma^t) = \left(\frac{1}{3}\right)^{\sum_{\tau=1}^{t} I_{s_{\tau}}} \left(\frac{2}{3}\right)^{\sum_{\tau=1}^{t} I_{b_{\tau}}}
= \left(\frac{1}{3}\right)^{\min_{a,b}{\{\sum_{\tau=1}^{t} I_{s_{\tau}}\}}} \left(\frac{2}{3}\right)^{\sum_{\tau=1}^{t} I_{b_{\tau}} - \min_{a,b}{\{\sum_{\tau=1}^{t} I_{s_{\tau}}\}}}
= \begin{cases} 
\left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{1}{3}\right)^{|S_t|} & \text{if } S_t > 0 \\
\left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{3}{3}\right)^{|S_t|} & \text{if } S_t < 0 
\end{cases}
\]

Since we are focusing on the \( \lim \sup \) of the likelihood ratio we can assume WLOG that \( S_t > 0 \), so that

\[
\left\{ \begin{array}{ll}
\ln p^{1}(\sigma^t) = \ln \left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{1}{3}\right)^{|S_t|}
\ln p^{FLS}(\sigma^t) = \ln \max_{i \in \{1,2\}} p^{i}(\sigma^t) - L(t) \ln \frac{4}{3} = \ln p^{2}(\sigma^t) - L(t) \ln \frac{4}{3} = \left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{2}{3}\right)^{|S_t|} - L(t) \ln \frac{4}{3}.
\end{array} \right.
\]

\[
\lim \sup \frac{p^{FLS}(\sigma^t)}{p^{1}(\sigma^t)} = \lim \sup \ln \frac{\max_{i \in \{1,2\}} p^{i}(\sigma^t)}{p^{1}(\sigma^t)} - L(t) \ln \frac{4}{3} \\
= \lim \sup \ln \frac{\left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{2}{3}\right)^{|S_t|}}{\left(\frac{2}{3}\right)^{\frac{1-|S_t|}{2}} \left(\frac{2}{3}\right)^{|S_t|}} - L(t) \ln \frac{4}{3} \\
= \lim \sup |S_t| \ln 2 - L(t) \ln \frac{4}{3} = +\infty \text{ } P\text{-a.s., By Lemma 2}
\]

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References


